Just What is Waving in Matter Waves?

- For waves in an ocean, it’s the water that “waves”
- For sound waves, it’s the molecules in medium
- For light it’s the \( E \) & \( B \) vectors
- What’s waving for matter waves?
  - It’s the PROBABILITY OF FINDING THE PARTICLE that waves!
  - Particle can be represented by a wave packet in
    - Space
    - Time
    - Made by superposition of many sinusoidal waves of different \( \lambda \)
    - It’s a “pulse” of probability

Imagine Wave pulse moving along a string: its localized in time and space (unlike a pure harmonic wave)
What Wave Does Not Describe a Particle

- What wave form can be associated with particle’s pilot wave?
- A traveling sinusoidal wave? \( y = A \cos (kx - \omega t + \Phi) \)
- Since de Broglie “pilot wave” represents particle, it must travel with same speed as particle ……(like me and my shadow)

\[
y = A \cos (kx - \omega t + \Phi)
\]

\[
k = \frac{2\pi}{\lambda}, \quad w = 2\pi f
\]

Conflicts with Relativity \( \Rightarrow \) Unphysical

Single sinusoidal wave of infinite extent does not represent particle localized in space

In Matter:

\[
\lambda = \frac{h}{p} = \frac{h}{\gamma mv}
\]

\[
f = \frac{E}{h} = \frac{\gamma mc^2}{h}
\]

\[\Rightarrow v_p = \lambda f = \frac{E}{p} = \frac{\gamma mc^2}{\gamma mv} = \frac{c^2}{v} > c!
\]
Wave Group or Wave Pulse

• Wave Group/packet:
  – Superposition of many sinusoidal waves with different wavelengths and frequencies
  – Localized in space, time
  – Size designated by
    • $\Delta x$ or $\Delta t$
  – Wave groups travel with the speed $v_g = v_0$ of particle

• Constructing Wave Packets
  – Add waves of diff $\lambda$,
  – For each wave, pick
    • Amplitude
    • Phase
  – Constructive interference over the space-time of particle
  – Destructive interference elsewhere!

Imagine Wave pulse moving along a string: its localized in time and space (unlike a pure harmonic wave).
Making Wave Packets: Simple Model with 2 waves

Ex: Phenomenon of "Beating" in Sound:

Add two waves of slightly different $\lambda$, $f$

$\Rightarrow$ Wave with: $f_{ave} = \left(\frac{f_1 + f_2}{2}\right)$, Amplitude $A \propto \left(\frac{f_1 - f_2}{2}\right)$

Start with two waves

$y_1 = ACos(k_1x - w_1t), \quad y_2 = ACos(k_2x - w_2t) : k = \frac{2\pi}{\lambda}, w = 2\pi f$
Resulting wave's "displacement" $y = y_1 + y_2$:

$$y = A\left[\cos(k_1x - w_1t) + \cos(k_2x - w_2t)\right]$$

Trigonometry: $\cos A + \cos B = 2\cos \left(\frac{A+B}{2}\right)\cos \left(\frac{A-B}{2}\right)$

$\therefore y = 2A\left[\left(\cos \left(\frac{k_2-k_1}{2}x - \frac{w_2-w_1}{2}t\right)\right)\left(\cos \left(\frac{k_2+k_1}{2}x - \frac{w_2+w_1}{2}t\right)\right)\right]$

since $k_2 \approx k_1 \approx k_{ave}$, $w_2 \approx w_1 \approx w_{ave}$, $\Delta k \ll k$, $\Delta w \ll w$

$\therefore y = 2A\left[\left(\cos \left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right)\right)\cos (kx - wt)\right] \equiv y = A' \cos (ks - wt)$, $A'$ oscillates in $x,t$

$A' = 2A\left(\cos \left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right)\right) = \text{modulated amplitude}$

**Phase Vel**

$$V_p = \frac{w_{ave}}{k_{ave}}$$

**Group Vel**

$$V_g = \frac{\Delta w}{\Delta k}$$

$$V_g : \text{Vel of envelope} = \frac{dw}{dk}$$
• Finite # of diff. Monochromatic waves always produce INFINTE sequence of repeating wave groups → can’t describe (localized) particle
• To make localized wave packet, add “infinite” # of waves with Well chosen Ampl A, Wave# k, ang.

\[ \psi(x, t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - wt)} dk \]

\( A(k) = \) Amplitude Fn
  \( \Rightarrow \) diff waves of diff k
  have different amplitudes \( A(k) \)
\( w = w(k), \) depends on type of wave, media

Group Velocity \( V_g = \frac{d\omega}{dk} \bigg|_{k=k_0} \)
Group, Velocity, Phase Velocity and Dispersion

In a Wave Packet: \( w = w(k) \)

Group Velocity \( V_g = \left. \frac{dw}{dk} \right|_{k=k_0} \)

Since \( V_p = wk \) \( (def) \Rightarrow w = kV_p \)

\[
V_g = \left. \frac{dw}{dk} \right|_{k=k_0} = V_p \left|_{k=k_0} + k \left. \frac{dV_p}{dk} \right|_{k=k_0}
\]

usually \( V_p = V_p(k \ or \ \lambda) \)

Material in which \( V_p \) varies with \( \lambda \) are said to be Dispersive

Individual harmonic waves making a wave pulse travel at different \( V_p \) thus changing shape of pulse and become spread out

In non-dispersive media, \( V_g = V_p \)

In dispersive media \( V_g \neq V_p \), depends on \( \frac{dV_p}{dk} \)

1ns laser pulse disperse
By x30 after travelling
1km in optical fiber
Matter Wave Packets

Consider An Electron:

mass = m  velocity = v,  momentum = p

Energy E = hf = γmc²;  ω = 2πf = \frac{2π}{h} γmc²

Wavelength λ = \frac{h}{p};  k = \frac{2π}{λ} \Rightarrow k = \frac{2π}{h} γmv

Group Velocity: V_g = \frac{dw}{dk} = \frac{dw}{dv} / \frac{dv}{dk}

\frac{dw}{dv} = \frac{d}{dv} \left[ \frac{2πmc^2}{h} \right] \left[ 1-(\frac{v}{c})^2 \right]^{1/2} = \frac{2πmv}{h \left[ 1-(\frac{v}{c})^2 \right]^{3/2}}

\frac{dk}{dv} = \frac{d}{dv} \left[ \frac{2π}{h} \frac{mν}{\left[ 1-(\frac{v}{c})^2 \right]^{1/2}} \right] = \frac{2πm}{h \left[ 1-(\frac{v}{c})^2 \right]^{3/2}}

V_g = \frac{dw}{dk} = \frac{dw}{dv} / \frac{dv}{dk} = v \Rightarrow Group velocity of electron Wave packet "pilot wave"

is same as electron's physical velocity

But velocity of individual waves making up the wave packet  \( V_p = \frac{w}{k} = \frac{c^2}{v} > c \)  (not physical)
Wave Packets & Uncertainty Principle

- Distance $\Delta X$ between adjacent minima $= (X_2)_{\text{node}} - (X_1)_{\text{node}}$
- Define $X_1 = 0$ then phase diff from $X_1 \rightarrow X_2 = \pi$

Node at $y = 0 = 2A \cos \left( \frac{\Delta w}{2} t - \frac{\Delta k}{2} x \right)$

$\Rightarrow \Delta k.\Delta x = \pi \Rightarrow$ Need to combine more $k$ to make small $\Delta x$ packet

Also implies $\Rightarrow \Delta p.\Delta x = h/2$

and

$\Delta w.\Delta t = \pi \Rightarrow$ Need to combine more $\omega$ to make small $\Delta t$ packet

Also $\Rightarrow \Delta E.\Delta t = h/2$

Amplitude Modulation

$y = 2A \left[ \cos \left( \frac{\Delta k}{2} x - \frac{\Delta w}{2} t \right) \right] \cos(kx - wt)$

What does this mean?