Some Relevant Formulae, Constants and Identities

\[ \lambda = \frac{h}{p} ; \quad \Delta x \Delta p \geq \frac{h}{4\pi} ; \quad \Delta E \Delta t \geq \frac{h}{4\pi} \]

Time Dep. S. Eq: \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t} \]

Time Indep. S. Eq: \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + U(x)\psi(x) = E \psi(x) \]

\[ [\hat{p}] = \frac{\hbar}{i} \frac{d}{dx} ; \quad [p^2] = -\hbar^2 \frac{\partial^2}{\partial x^2} ; \quad [\hat{K}] = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} ; \quad [\hat{E}] = \hbar \frac{\partial}{\partial t} \]

What to expect when expecting: \[ \langle Q \rangle = \int_{-}\Psi^*(x,t)[\hat{Q}]\Psi(x,t)dx \]

Uncertainty in Observable \( Q \): \[ \Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2} \]

Quantum Oscillator in Ground State \( \psi(x) = \frac{m\omega}{\pi \hbar} \sqrt{\frac{1}{2n!}} e^{-\frac{m}{2\hbar}\omega x^2} \]

Energy of Quantum Oscillator \( E_n = (n + \frac{1}{2})\hbar \omega \)
Problem 1: Go Jump Off A Quantum Cliff [12 pts]:
A beam of particles, each with energy \( E > 0 \) is incident from the left (of \( x = 0 \)) on to a potential described by
\[
\begin{align*}
U(x) &= 0 \quad \text{for region I} \quad (x < 0) \\
U(x) &= -V_0 \quad \text{for region II} \quad (x > 0)
\end{align*}
\]
(a) sketch the potential as a function of \( x \). (b) Write down the expression for the time independent Schrodinger Eqn in each region I and II in terms of wavenumbers \( k = \sqrt{\frac{2m(E - U(x))}{\hbar}} \). (c) Find the general expression for \( \psi_i(x) \) & \( \psi_f(x) \) in terms of \( k_1 \) & \( k_2 \) respectively. (d) Reviewing the physics of the situation which component of \( \psi_f(x) \) can be thrown out? Sketch the wave function in the two regions. Is the particle wavelength the same in the two regions? (e) Using the continuity conditions for \( \psi \) & \( \frac{d\psi}{dx} \), calculate the expression for relative rates at which the particles are reflected (R) and transmitted (T) across the potential step in terms of \( k_1 \) & \( k_2 \). (f) Is the reflection rate the same in the quantum picture as in the classical picture? Explain the difference.

Problem 2: “Lazy ‘R Us” Is The Physics Mantra [8 pts]:
Consider a quantum Harmonic oscillator, of mass \( m \) under potential \( U(x) = \frac{1}{2} m \omega^2 x^2 \), in its ground state. For such a system you learnt in homework that \( \langle x \rangle = 0 \) and \( \langle x^2 \rangle = \frac{\hbar}{2m\omega} \). (a) Calculate the uncertainty \( \Delta x \) in its location \( x \). (b)
Now estimate \( \langle p \rangle \). (c) Write the expression for the total non-relativistic energy for this system and use it to relate \( \langle p^2 \rangle \) to \( \langle x^2 \rangle \). (d) Finally, calculate the uncertainty \( \Delta p \) and the value of the product \( \Delta x \cdot \Delta p \). How well does your calculation agree with Heisenberg’s Uncertainty relation?

“Mantra”: A sacred word repeated in prayer or meditation.
Phys 2D Quiz 8 Solns

\[ u(x) \]

\[ a \]

\[ -V_0 \]

\[ b \]

\[ k_I = \sqrt{\frac{2mE}{\hbar}} \quad k_{II} = \sqrt{\frac{2m(E + V_0)}{\hbar}} \]

So Region I: \[-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E \psi \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -k_I^2 \psi \]

Region II: \[-\frac{\hbar^2}{2m} \frac{\partial^4 \psi}{\partial x^4} - V_0 \psi = E \psi \Rightarrow \frac{\partial^4 \psi}{\partial x^4} = -k_{II}^2 \psi \]

\[ c \]

\[ \psi_I = A e^{ik_I x} + B e^{-ik_I x} \]

\[ \psi_{II} = C e^{ik_{II} x} + D e^{-ik_{II} x} \]

\[ \text{Can lose the } D e^{-ik_{II} x} \text{ component, since nothing's reflected in } II. \]

The wavelength is not the same in the 2 regions - since \( k_{II} > k_I \), \( \lambda_{II} < \lambda_I \).

Sketch: \[ \text{Wavy line} \]
Continuous: \( \psi(0) = \psi(0) \Rightarrow A + B = C \)

\[
\frac{\partial \psi}{\partial x} \bigg|_{x=0} = \frac{\partial \psi}{\partial x} \bigg|_{x=0} \Rightarrow iK_A A - iK_B B = iK_C C
\]

So \( B = \frac{K_A - K_B}{K_A + K_B} A, \quad C = \frac{2K_B}{K_A + K_B} A \)

\[
R = \frac{|B|^2}{|A|^2} = \frac{(K_A - K_B)^2}{(K_A + K_B)^2}
\]

\[
T = 1 - R = \frac{4K_A K_B}{(K_A + K_B)^2}
\]

\( \boxed{\text{No, in the classical picture } R = 0!} \)

\( \text{So not even sorta the same.} \)

2.

a) \( \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \left( \frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} \)

b) \( \langle p \rangle = 0 \) since the thing is oscillating.

Could calculate this too.

\[ \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 = E, \text{ so } \langle p^2 \rangle = 2m \langle E \rangle - \frac{1}{2} m \omega^2 \langle x^2 \rangle \]

2)

\[ \langle p^2 \rangle = 2m E - m^2 \omega^2 \frac{\hbar^2}{2m^2} = 2m \frac{1}{2} k_0 - \frac{1}{2} \hbar \omega = \frac{1}{2} \hbar \omega \]

So \( \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \left( \frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} \Rightarrow \Delta x \Delta p = \frac{\hbar}{2} \]

Agrees with HUP! (It's the minimum uncertainty.)