Special Relativity

One of the biggest surprises in our understanding of Physics
Extend Inertial Coordinate System
Symmetry of Galileo and Newton

- THE EINSTEIN PRINCIPLE OF RELATIVITY
  - "ALL OF THE LAWS OF PHYSICS are the same for every inertial observer."
- In particular,
  - "The speed of light is the SAME for all inertial observers, regardless of the motion of the source."
- But this required some big changes in our basic understanding of time.
The speed of light \( c \) was the central question that gave rise to the theory of relativity.

The speed of light is very large compared to the speeds we experience.

- We have no physical intuition about speeds approaching \( c \).

We can however measure the speed of light with a rotating mirror.

- Light bounces off the mirror twice with a long distance in between
- A small deflection results:
  \[
  \delta x = \frac{8\pi DD' f}{c} \approx \frac{25(10^2)}{3 \times 10^8 \text{ m/s}} \times \frac{600 \text{ / s}}{10^8 \text{ m/s}} = 0.005 \text{ m}
  \]
In what frame is the speed of light $c$?

- The equations of EM were consistent if the speed of light is constant in one fixed frame.
  - Physicists thought EM waves must propagate in some medium.
  - Postulated the “ether” (aether).
    - They thought, space is filled with “the ether” in which EM waves propagate at a fixed speed.
  - Ether gave one fixed frame for EM.

- But experiments disagreed.
- And we would lose the symmetry found in Newton’s laws; “any inertial frame”.
The Sun and Earth are Moving

- We should see some velocity of the ether.
- We should see a seasonal variation.
  - MM set up to be sensitive even to the motion of the earth.
The Michelson-Morley Experiment

Albert Abraham Michelson (1852-1931) was a German-born U.S. physicist who established the speed of light as a fundamental constant. He received the 1907 Nobel Prize for Physics. In 1878 Michelson began work on the passion of his life, the measurement of the speed of light. His attempt to measure the effect of the earth's velocity through the supposed ether laid the basis for the theory of relativity. First American scientist to win Nobel Prize.

Edward Williams Morley (1838-1923) was an American chemist whose reputation as a skilled experimenter attracted the attention of Michelson. In 1887 the pair performed what has come to be known as the Michelson-Morley experiment to measure the motion of the earth through the ether.
Michelson-Morley Experiment

- Michelson interferometer on a block of granite.
  - Light waves from two mirrors will interfere.
- Granite block floats in mercury to greatly reduce vibration.
- Slowly rotate apparatus and measure interference.
  - MM found no change as they rotated.
  - Speed of light is the same even though the earth is moving.

Wavelength of light is about 400 nm so interferometer measurement is very accurate.
MM calculation with ether

- Ether with just the earth’s velocity would show interference.
- As apparatus is rotated, interference changes.

\[ t_1 = \frac{L'}{c} + \frac{L'}{c} \quad \text{in direction perpendicular to ether motion} \]

\[ L' = \sqrt{L^2 + (vt_1)^2} \approx L\sqrt{1 + \frac{v^2}{c^2}} \]

\[ t_2 = \frac{L}{c + v} + \frac{L}{c - v} \quad \text{in direction parallel to ether motion} \]

\[ t_2 - t_1 = \frac{L(c-v)}{c^2-v^2} + \frac{L(c+v)}{c^2-v^2} - \frac{2L'}{c} = \frac{2L}{c} - \frac{2L'}{c} = \frac{2L}{c} \sqrt{1 + \frac{v^2}{c^2}} - \frac{2L}{c} \left( 1 + \frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2} \right) \approx \frac{2L}{c} \left( \frac{1 + v^2}{c^2} - \frac{v^2}{2c^2} \right) = \frac{L}{c} \frac{v^2}{c^2} \]

\[ c(t_2 - t_1) = (11m) \left( \frac{3 \times 10^4}{3 \times 10^8} \right)^2 = 11 \times 10^{-8} m = 110 nm \quad \text{a large fraction of the wavelength of light} \]
Heaviside, Lorentz, Fitzgerald

Working on Electrodynamics of moving bodies

Oliver Heaviside (1850-1925) was a telegrapher, but deafness forced him to retire and devote himself to investigations of electricity. He became an eccentric recluse, befriended by Fitzgerald and (by correspondence) by Hertz. In 1892 he introduced the operational calculus (Laplace transforms) to study transient currents in networks and theoretical aspects of problems in electrical transmission. In 1902, after wireless telegraphy proved effective over long distances, Heaviside theorized that a conducting layer of the atmosphere existed that allows radio waves to follow the Earth’s curvature. He invented vector analysis and wrote Maxwell’s equations as we know them today. He showed how EM fields transformed to new inertial frames.

Hendrik Antoon Lorentz (1853-1928), a professor of physics at the University of Leiden, sought to explain the origin of light by the oscillations of charged particles inside atoms. Under this assumption, a strong magnetic field would effect the wavelength. The observation of this effect by his pupil, Zeeman, won a Nobel prize for 1902 for the pair. However, the Lorentz theory could not explain the results of the Michelson-Morley experiment. Influenced by the proposal of Fitzgerald, Lorentz arrived at the formulas known as the Lorentz transformations to describe the relation of mass, length and time for a moving body. These equations form the basis for Einstein’s special theory of relativity.

Einstein read Lorentz’s book

George Francis Fitzgerald (1851-1901), a professor at Trinity College, Dublin, was the first to suggest that an oscillating electric current would produce radio waves, laying the basis for wireless telegraphy. In 1892 Fitzgerald suggested that the results of the Michelson-Morley experiment could be explained by the contraction of a body along its direction of motion.

Einstein
“On the Electrodynamics of Moving Bodies”

• Einstein had read Lorentz’s book and worked for a few years on the problem.
• He did not believe there should be one fixed frame.
• He had a breakthrough, “the step” in 1905.

While Einstein didn’t read much about the MM experiment, Lorentz did.
Albert Einstein (1879-1955), one of the great geniuses of physics, grew up in Munich where his father and his uncle had a small electrical plant and engineering works. Einstein's special theory of relativity, first printed in 1905 with the title "On the Electrodynamics of Moving Bodies" had its beginnings in an essay Einstein wrote at age sixteen. The special theory is often regarded as the capstone of classical electrodynamic theory.

Einstein did not get a Nobel prize for SR. He got one for contributions to theoretical physics including the photoelectric effect. The committee did not think SR had been proved correct until the 1940s.

Assistant patent agent, third class Einstein
Velocity Addition

• Einstein wanted the speed of light to be the same in every frame.
  o This would work for EM equations.
  o It would agree with experiment.
    → Einstein did consider experiment but maybe not Michelson Morley.

• But velocity addition didn’t make sense to anyone.
  o How could an observer in an inertial frame moving at 0.9c measure light to move at the same speed as we do in our frame at rest?

\[
x' = x + 0.9ct
\]

\[
v' = \frac{dx'}{dt} = \frac{dx}{dt} + 0.9c = \nu + 0.9c
\]
“The Step”

- Einstein realized that by **discarding the concept of a universal time**, the speed of light could be the same in every frame.
  - In going from one inertial frame to another, both $x$ and $t$ transform.
  - The time is different in different inertial frames of reference.
  - He derived the previously stated Lorentz transformation from the requirement that the speed of light is the same in every inertial frame.

\[
x' = \gamma (x - vt) \\
t' = \gamma \left( t - \frac{v}{c^2} x \right) \\
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]
Minkowski Space (1907)

- Hermann Minkowski proposed that special relativity could be best expressed in 4 dimensions $x_\mu = (x, y, z, t)$, 4-vector has a greek subscript
  - With a new and unusual dot product
  - (Important that $c=1$ or else $t \rightarrow ct$)
    \[
    a_\mu \cdot b_\mu = a_x b_x + a_y b_y + a_z b_z - a_t b_t
    \]
- We do not notice this geometry because we move at very small angles from the $t$ axis.
  \[
  \text{angle} = \frac{\Delta x}{\Delta t} = \frac{v}{c} \ll 1 \quad \text{for humans}
  \]
Future, past and light-cone

- Humans move almost directly along the time axis.
  - Photons move on the light-cone
  - We can shoot other particles close to the light-cone
- Spacelike vectors have positive norm
  - (Dot product with itself)
- Timelike vectors have negative norm.
- Lightlike vectors are on the light-cone and have zero norm.
Example: Norm of a 4-vector

• Dot the 4-vector with itself

\[ \vec{x} = (x, y, z, ct) = (1, 2, 3, 4) \]
\[ \text{norm} = \vec{x} \cdot \vec{x} = 1 + 4 + 9 - 16 = -2 \]
this is a "future pointing" timelike vector

• Norm>0, spacelike
• Norm<0, timelike
• Norm=0, lightlike
4-vectors

• 4-vectors transform under rotations
• The dot product of 4-vectors is a scalar and is invariant under rotations
• The norm of a vector can be positive, negative or zero
  o Spacelike, timelike or lightlike respectively
• Transforming to an inertial frame moving in the x direction is a rotation in the x-t plane.
Rotations in 4D

Rotation in the $xy$ plane.

$$a_{\mu\nu} = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

All 4-vectors transform the same way.

Rotation in the $xt$ plane

$$a_{\mu\nu} = \begin{pmatrix} \cos i\theta & 0 & 0 & \sin i\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin i\theta & 0 & 0 & \cos i\theta \end{pmatrix} = \begin{pmatrix} \cosh \theta & 0 & 0 & i \sinh \theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i \sinh \theta & 0 & 0 & \cosh \theta \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta \gamma & 0 & 0 & \gamma \end{pmatrix}$$

A rotation in the $xt$ plane is a “boost” along the $x$ direction to another inertial frame.

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\cosh \theta = \gamma$$

$$\tanh \theta = \beta$$

$$\sinh \theta = \beta \gamma$$
Rotation Symmetry in 4D

• Includes 3D rotations
  o Rotations in $xy$, $yz$, $xz$ planes
• Plus symmetry that all the laws of physics are the same in any inertial frame.
  o Rotation in $xt$ plane is boost in the $x$ direction
  o Rotation in $yt$ plane is boost in the $y$ direction
  o Rotation in $zt$ plane is boost in the $z$ direction
• All 4-vectors transform with the same matrix
Question: Michelson Morley

- Which of the following statements is false?

A) MM compares the speed of light in perpendicular directions.

B) MM should have detected the ether had it existed.

C) MM used the interference of light waves.

D) MM tends to show that the speed of light is a constant independent of velocity of the observer.

E) None of the above.
Effects of Lorentz Transformation

• Time dilation
  o A fast moving particle lives much longer in the lab than in its rest frame.

• Lorentz contraction
  o Objects are shorter in the lab frame in which they are moving than they are in their rest frame.

\[ x' = \gamma (x - vt) \]

\[ t' = \gamma \left( t - \frac{v}{c^2} x \right) \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]
Energy Momentum 4-vector

\[ x_\mu = (x, y, z, ct) \]

keep \( c \) in case \( c \neq 1 \)

\[ p_\mu = \left( p_x, p_y, p_z, \frac{E}{c} \right) \]

scalar giving distance to event

\[ x_\mu \cdot x_\mu \]

\[ p_\mu \cdot p_\mu = p^2 - \frac{E^2}{c^2} = -m^2 c^2 \]

scalar mass

\[ E = mc^2 \]

when particle is at rest

\[ p_\mu \cdot x_\mu = \vec{p} \cdot \vec{x} - Et \]

phase in Quantum Mechanics
Special Relativity (1905-1908)

• Laws of physics are independent of which inertial frame of reference we choose (Newtonian mechanics).
• Extend this to speed of light.
  o Speed of light the same in any inertial frame.
  o Michelson Morley experiment.
• It turns out we really live in 4 Dimensions
  o 3 space, 1 time
  o Symmetry under rotations in 4D
Spark Chamber

- It was felt (at least by the Nobel committee) that there was no good experimental evidence in favor of Special Relativity.
- Some of the first clear evidence was from the muons produced by cosmic rays.
  - While muons have a lifetime measured in their rest frame of about 2 microseconds,
  - They can travel 100 km without decaying
A Boost in the x Direction

\[(x, y, z, t)\] initial coordinates in 4D

transform to inertial system moving with velocity \(v\) along \(x\) axis

\[(x', y', z', t')\] transformed coordinates in 4D

\[x' = \gamma (x - vt)\] Lorentz Transformation

\[y' = y\]

\[z' = z\]

\[t' = \gamma \left(t - \frac{v}{c^2} x\right)\] Lorentz Transformation

\[\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\] depends on \(v\)
Example: Time Dilation

Assume the muon lifetime is $\tau$. If it has a velocity of $0.999c$, how far does it travel before it decays?

start in a frame at rest with the muon
the muon is at the origin
assume the muon decays at time $\tau$

$$r = \sqrt[3]{\frac{\tau^2 GM}{4\pi^2}} \approx 4.2 \times 10^7 \text{ m}$$

$$r = 42000 \text{ km}$$

$x' = \gamma(x - vt)$
$t' = \gamma\left(t - \frac{v}{c^2} x\right)$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$
Problem: Muon Lifetime

- The mean muon lifetime is about 2 microseconds. Without the time dilation of SR, how far would a muon moving at a velocity of 0.999c typically travel before it decays?

\[
x' = \gamma (x - vt)
\]
\[
t' = \gamma \left( t - \frac{v}{c^2} x \right)
\]
\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

A) 6 m  
B) 60 m  
C) 600 m  
D) 6000 m  
E) Not enough info to answer.
Problem: Muon Lifetime

- The mean muon lifetime is about 2 microseconds. Without the time dilation of SR, how far would a muon moving at a velocity of 0.99c typically travel before it decays?

\[
\begin{align*}
x &= vt + x_0 \\
x' &= \gamma(x - vt) \\
t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\
\gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
\end{align*}
\]

Options:

A) 6 m
B) 60 m
C) 600 m
D) 6000 m
E) Not enough info to answer.
**Tough Problem: Muon Lifetime**

- The mean muon lifetime is about 2 microseconds. According to SR, how far would a muon moving at a velocity of 0.999c typically travel before it decays?

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>A)</td>
<td>13000 m</td>
</tr>
<tr>
<td>B)</td>
<td>1300 m</td>
</tr>
<tr>
<td>C)</td>
<td>130 m</td>
</tr>
<tr>
<td>D)</td>
<td>13 m</td>
</tr>
<tr>
<td>E)</td>
<td>1.3 m</td>
</tr>
</tbody>
</table>
Tough Problem: Muon Lifetime

- The mean muon lifetime is about 2 microseconds. According to SR, how far would a muon moving at a velocity of 0.9999c typically travel before it decays?

\[
x = vt + x_0
\]
\[
x' = \gamma (x - vt)
\]
\[
t' = \gamma \left( t - \frac{v}{c^2} x \right)
\]
\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

- A) 13000 m
- B) 22000 m
- C) 32000 m
- D) 42000 m
- E) 130000 m
End Quiz 3
\[ F = \frac{GMm}{r_{12}^2} \]
\[ a = \frac{\nu^2}{r} \]
\[ \tau = \frac{2\pi r}{\nu} = \frac{2\pi r^{3/2}}{\sqrt{GM}} \]
\[ G = 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \]
\[ PE = - \frac{GMm}{r} \]
\[ KE = \frac{1}{2} mv^2 \]
\[ E = KE + PE \]
\[ r_{\text{earth-orbit}} = 1.5 \times 10^{11} \text{ m} \]
\[ m_{\text{earth}} = 6 \times 10^{24} \text{ kg} \]

\[ a_\mu \cdot b_\mu = a_x b_x + a_y b_y + a_z b_z - a_t b_t \]
\[ x_\mu = (x, y, z, ct) \]
\[ p_\mu = \left( p_x, p_y, p_z, \frac{E}{c} \right) \]
\[ x_\mu \cdot x_\mu = s \]
\[ p_\mu \cdot p_\mu = p^2 - \frac{E^2}{c^2} = -m^2 c^2 \]
\[ E = \sqrt{(pc)^2 + (mc^2)^2} \]
\[ E_{\text{rest-frame}} = mc^2 \]
\[ p_\mu \cdot x_\mu = \vec{p} \cdot \vec{x} - Et \]

\[ x' = \gamma (x - vt) \]
\[ t' = \gamma \left( t - \frac{v}{c^2} x \right) \]
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ c = 3 \times 10^8 \text{ m} \text{ s}^{-1} \]

The gravity of the visible matter in the Galaxy is not enough to explain the high orbital speeds of stars in the Galaxy. For example, the Sun is moving about 250 kilometers per second. The part of the rotation curve contributed by the visible matter only is the bottom curve. The discrepancy between the two curves is evidence for a dark matter halo.
Question: Units

To make the laws of physics simple and to make rotations in 4D most easily understandable, what should the speed of light be?

- A) $3 \times 10^8$ m/s
- B) $3 \times 10^{10}$ cm/s
- C) 1 m/s
- D) 1 cm/s
- E) 1
Example: Distance in 4D

• Assume an observer is at rest at the origin. If the coordinates of an event in that frame are \((x_e, y_e, z_e, cte)\) meters, at what time did/will he first see the light from that event?

light has to travel a distance of

\[ d = \sqrt{x_e^2 + y_e^2 + z_e^2} \]

the event happened at time

\[ t_e \]

light should then arrive at

\[ t = t_e + \frac{d}{c} \]

we could use the spacetime interval

\[ t - t_e = \frac{\sqrt{x_e^2 + y_e^2 + z_e^2}}{c} \]

\[ ct - cte = \sqrt{x_e^2 + y_e^2 + z_e^2} \]

\[ (ct - cte)^2 = x_e^2 + y_e^2 + z_e^2 \]

light interval is zero

\[ s = x_e^2 + y_e^2 + z_e^2 - (ct - cte)^2 = 0 \]
Problem: Distance in 4D

\[ x' = \gamma (x - vt) \]
\[ t' = \gamma \left( t - \frac{v}{c^2} x \right) \]
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ s = x_e^2 + y_e^2 + z_e^2 - (ct - ct_e)^2 \]

- Assume an observer is at rest at the origin. If the coordinates of an event in that frame are \((x, y, z, ct) = (1500, 0, 0, 1500)\) m, at what time did/will he first see the light from that event?

A) \(t = -5\) msec
B) \(t = -5\) \(\mu\)sec
C) \(t = 10\) msec
D) \(t = 10\) \(\mu\)sec
E) \(t = 5\) \(\mu\)sec
Problem: Distance in 4D

\[ x' = \gamma(x - vt) \]
\[ t' = \gamma\left(t - \frac{v}{c^2}x\right) \]
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ s = x_e^2 + y_e^2 + z_e^2 - (ct - ct_e)^2 \]

- Assume an observer is at rest at the origin. If the coordinates of an event in that frame are \( (x, y, z, ct) = (0, 6000, 0, -1500) \) m, at what time did/will he first see the light from that event?

   A) \( t=-15 \) µsec
   B) \( t=5 \) µsec
   C) \( t=10 \) µsec
   D) \( t=15 \) µsec
   E) \( t=25 \) µsec
Question: Minkowski Space

• Assume an observer is at rest at the origin and \( t=0 \). If the coordinates of an event in that frame are \((x,y,z,ct)=(500,0,0,-1500)\) m, where does the event appear in the observer’s space-time diagram?

<table>
<thead>
<tr>
<th>Answer Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>A) future</td>
</tr>
<tr>
<td>B) past</td>
</tr>
<tr>
<td>C) On Future light cone</td>
</tr>
<tr>
<td>D) On Past light cone</td>
</tr>
<tr>
<td>E) Causally disconnected</td>
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</tbody>
</table>

Our path through space-time is nearly on time axis.
Energy and Momentum

\[ p_\mu c = (p_x c, p_y c, p_z c, E) \]
\[ (p_\mu c) \cdot (p_\mu c) = (pc)^2 - E^2 = -(mc^2)^2 \] scalar mass
\[ E^2 = (pc)^2 + (mc^2)^2 \]
\[ E = mc^2 \] in rest frame
\[ p_\mu c = (0, 0, 0, mc^2) \] in rest frame
\[ p_\mu c = (\beta \gamma mc^2, 0, 0, \gamma mc^2) \] boosting in x direction
\[ pc = \beta \gamma mc^2 \]
\[ E = \gamma mc^2 \]
Problem: Mass into Energy

- If we could convert $10^{-3}$ kg of matter into energy, how fast could we make a 1000 kg car go?

\[
P.E. = mgh
\]
\[
K.E. = \frac{1}{2}mv^2
\]
\[
E = mc^2
\]

A) 420000 m/s
B) 13000 m/s
C) 130 m/s
D) 8 m/s
E) 0.8 m/s
Space-Time

• We see the universe expanding
• We can look back in time to see the early universe
• The universe is about 13.6 billion years old
• Space-Time itself was created in the big bang
Special Relativity

- Minkowski space (3 space, 1 time)
- Laws of physics invariant under translations and rotations in Minkowski space
- Boost to a new inertial frame of reference is just a rotation in Minkowski space!
  - To see this, it's important to get the units right (c=1)
- Some vectors in Minkowski space
  - (x, y, z; t)
  - (p_x, p_y, p_z; E)
- This is all there is to Special Relativity.
Subsequent Developments

• By now, SR, MM are extremely well tested.
  o Don’t believe crackpots on the web.
• Einstein went on to use curved 4D spacetime to develop GR, a new theory of gravity that is tested fairly well too.
General Relativity

- General Relativity puts us in curved space-time.
- Gravity is equivalent to acceleration.
- It is a well tested theory of gravity with new phenomena
  - Black holes
  - Bending of light
  - Cosmology
- It is not easy to make a theory of Quantum Gravity
- Curvature of space-time is related to mass and energy density

Einstein Equation

\[ G^{\mu\nu} = 8\pi T^{\mu\nu} \]
Black Holes

• Light cannot escape a black hole (nothing can).
• From the outside, we see a collapse to $R=2M$ (in some units).
• It takes light an infinite amount of time to get out from $R=2M$.
• If you fall inside, the black hole collapses to a point!
  o But you get torn apart by tidal forces before you see much.
The Electromagnetic Field

Magic, Invisible force fields…

Ancient Greek’s observed rubbed amber attracting straw…

Electrum: Latin for amber
New Physics: E&M

• Mechanics was relatively simple.
  o Explained the motion of bodies
  o Didn’t explain existence of matter, light, elements, chemistry…
  o Didn’t explain rather minor effects of static electricity and magnetism.

• Studies of these effects changed the world and our understanding of Physics.

• They led to our understanding of all the areas listed above.
Static Electricity with Rods

- We can see the existence of charge rather simply.
- By rubbing some objects we can transfer charge between them leaving one object with positive charge and another with negative.
- Charge somehow creates Electric Field
- There is a force on a charge in an Electric Field.
Electrons

• An electron is a relatively light, fundamental particle with a charge of \(-1.60 \times 10^{-19}\) Coulombs.
• It is the particle that is transferred to charge up rods...
• Bound electrons determine the properties of atoms and molecules.
• It is the particle that moves in wires, transistors, ICs, radios, computers… to make those devices work.
Van de Graaff

• With a Van de Graaff generator we can really charge up an object.
• Electrons are moved from one object to another building up a large charge and a large electric field.
Van de Graaf

• Can produce a large voltage but not much current.
  o Current = charge per second.

• Need very dry conditions to do this.
Electric Field Lines

- We wish to see the nature of the Electric field.
- We can see the field lines.
  - Field lines start on positive charges and end on negative charges.
- It is a vector field
- The field lines are conserved, only beginning and ending on charges
Field Lines

- Field Lines start on + charges and end on - charges.
- It is a vector field.

\[
\vec{E}(\vec{x}, t) \quad \text{field is a function of position and time vector field}
\]

\[
\oint \vec{E} \cdot d\vec{S} = q_{\text{enclosed}} \quad \text{field strength (number of field lines) is proportional to charge field lines are continuous}
\]
What is a Field

• A field is a function of position and time.
• A vector field is a vector function of position and time.
  - \( \vec{E}(x, y, z, t) \)
  - \( \vec{B}(x, y, z, t) \)
• These fields are natural phenomena that we must understand.
We Live in an Electromagnetic World

• E field binds atoms together
  o Binds atoms into molecules
  o Metals, semiconductors…
  o Biology
  o (nuclear force determines what elements we have)

• EM waves
  o Light is our primary energy source
  o Radio waves
  o EM waves have both E and B fields.
Question: Fields

What properties do both the E and B fields have?

- A) They can depend on position and time.
- B) They are both vectors having magnitude and direction.
- C) They are present in EM radiation (light…).
- D) A and B.
- E) All of the above.

Answer: C

- C) They are present in EM radiation (light…).
Magnetic Field Lines in 2D and 3D

- There are also magnetic field lines.
- It's also a vector field.
- Magnetic field lines circulate around current
  - Current is a flow of electric charge
  - Moving charges generate loops of field

\[ \vec{B}(\vec{x}, t) \] field is a function of position and time
vector field
\[ \oint \vec{B} \cdot d\vec{S} = 0 \] as many field lines enter any volume as exit it
field lines are continuous
Maxwell’s Equations

Fields obey equations. These equations fully describe E&M.

\[ \nabla \cdot \vec{B} = 0 \]

\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \]

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

\[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \]

\[ \vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}) \]

Equations are fairly complex with 3 constants.
Maxwell’s Equations (1864)

• Were the result of many studies by Coulomb (1784), Ampere (1820), Faraday (1831), Gauss (1837) and others.
• Unified electricity and magnetism
• Predicted EM waves
• His 20 equations were later rewritten by Heaviside in vector notation as 4 equations (1887).
Static Charge Gives Static E-field

\[ \nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]

\[ \nabla \times \vec{E} + \frac{\partial B}{\partial t} = 0 \]

\[ \nabla \cdot \vec{B} = \frac{\rho}{\varepsilon_0} \]

\[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial E}{\partial t} = \mu_0 \vec{J} \]

\[ \vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}) \]

E-field lines start on positive charges,
And end on negative charges.
Field lines are continuous.
E-field is proportional to the number of field lines per unit area.
E Field of Point Charge
or Outside a Charged Sphere

\[ \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \]  
field of a point charge

\[ \vec{F} = q_2 \vec{E} \]  
force on a charge in a E-field

\[ -e = -1.60 \times 10^{-19} \text{ Coulombs} \]  
charge on an electron

\[ \varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N m}^2} = \frac{\text{C}^2 \text{s}^2}{\text{kg m}^3} \]  

The SI unit of charge is the Coulomb. All I can say is that it is defined in a stupid way.
Direction of the E-field

• The field of a point charge or a charged sphere points radially outward from the charge.

• We denote this by the unit vector in the radial direction $\hat{r}$

• We may use the unit vectors $\hat{x}, \hat{y}, \hat{z}$ to give the direction of vectors in Cartesian coordinates.
Problem: E-field

- If a small charged sphere with $q=3.0 \times 10^{-12}$ Coulombs is placed at the origin, what is the Electric field at $(x,y,z)=(0,0.2,0)$?

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}
\]
\[
\vec{F} = q_2 \vec{E}
\]
\[-e = -1.60 \times 10^{-19} \text{ C}\]
\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2\text{s}^2}{\text{kg}\text{m}^3}
\]

<table>
<thead>
<tr>
<th>Option</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>A)</td>
<td>$-0.67 \frac{\text{kg m}}{\text{s}^2 \text{C}} \hat{y}$</td>
</tr>
<tr>
<td>B)</td>
<td>$0.67 \frac{\text{kg m}}{\text{s}^2 \text{C}} \hat{x}$</td>
</tr>
<tr>
<td>C)</td>
<td>$-0.67 \frac{\text{kg m}}{\text{s}^2 \text{C}} \hat{z}$</td>
</tr>
<tr>
<td>D)</td>
<td>$-0.67 \frac{\text{kg m}}{\text{s}^2 \text{C}} \hat{x}$</td>
</tr>
<tr>
<td>E)</td>
<td>$0.67 \frac{\text{kg m}}{\text{s}^2 \text{C}} \hat{y}$</td>
</tr>
</tbody>
</table>
Problem: E-field

- If a small charged sphere with \( q = -3.0 \times 10^{-12} \) Coulombs is placed at the origin, what is the Electric field at \( (x,y,z) = (0,0,-0.5) \)?

\[
\vec{E} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{r}
\]
\[
\vec{F} = q \vec{E}
\]
\[
e = -1.60 \times 10^{-19} \text{ C}
\]
\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2 \text{s}^2}{\text{kg} \text{m}^3}
\]

<table>
<thead>
<tr>
<th>Option</th>
<th>Electric Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>A)</td>
<td>( 0.67 \frac{\text{kg} \text{m}}{\text{s}^2 \text{C}} \hat{z} )</td>
</tr>
<tr>
<td>B)</td>
<td>( 0.67 \frac{\text{kg} \text{m}}{\text{s}^2 \text{C}} \hat{y} )</td>
</tr>
<tr>
<td>C)</td>
<td>( -0.67 \frac{\text{kg} \text{m}}{\text{s}^2 \text{C}} \hat{z} )</td>
</tr>
<tr>
<td>D)</td>
<td>( -0.11 \frac{\text{kg} \text{m}}{\text{s}^2 \text{C}} \hat{z} )</td>
</tr>
<tr>
<td>E)</td>
<td>( 0.11 \frac{\text{kg} \text{m}}{\text{s}^2 \text{C}} \hat{z} )</td>
</tr>
</tbody>
</table>
Lorentz Force Equation

\[ \vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}). \]

Force on an electron with charge \(-e\)

Force is parallel to E-field

Force is perpendicular to B-field and \(\vec{v}\)
Example: Force between Point Charges

- Assume $q_1 = q_2 = 1 \mu \text{Coulomb}$
- at a distance of 0.2 m.

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r^2} \hat{r}
\]

Field of a point charge

\[
\vec{F} = q_2 \vec{E}
\]

Force on a charge in a E-field

\[-e = -1.60 \times 10^{-19} \text{ Coulombs}\]

\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2\text{s}^2}{\text{kg m}^3}
\]

\[
\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}
\]

Opposites attract

\[
\vec{F} = \frac{1}{4\pi} \left( \frac{10^{-6} \text{C}}{0.2 \text{m}} \right)^2 \hat{r} = \frac{1}{4\pi} \left( 8.85 \frac{s^2}{\text{kg m}} \right) (0.2)^2 \hat{r} = 0.22 \frac{\text{kg m}}{\text{s}^2}
\]
Problem: Force on point charge

- A charge \( q_1 = 3 \times 10^{-6} \text{ C} \) is located at the origin. What is the force on a charge \( q_2 = 5 \times 10^{-6} \text{ C} \) which is located at \((x, y, z) = (0, 0.05, 0) \text{ m}\)?

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}
\]

\[
\vec{F} = q_2 \vec{E}
\]

\[-e = -1.60 \times 10^{-19} \text{ C} \]

\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2 \text{s}^2}{\text{kg m}^3}
\]

<table>
<thead>
<tr>
<th>Option</th>
<th>Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>A)</td>
<td>(2.7 \frac{\text{kg m}}{\text{s}^2} \hat{x})</td>
</tr>
<tr>
<td>B)</td>
<td>(54 \frac{\text{kg m}}{\text{s}^2} \hat{x})</td>
</tr>
<tr>
<td>C)</td>
<td>(54 \frac{\text{kg m}}{\text{s}^2} \hat{y})</td>
</tr>
<tr>
<td>D)</td>
<td>(-54 \frac{\text{kg m}}{\text{s}^2} \hat{x})</td>
</tr>
<tr>
<td>E)</td>
<td>(-54 \frac{\text{kg m}}{\text{s}^2} \hat{y})</td>
</tr>
</tbody>
</table>
Problem: Force on point charge

- A charge \( q_1 = 2 \times 10^{-6} \text{ C} \) is located at the origin. What is the force on a charge \( q_2 = 5 \times 10^{-6} \text{ C} \) which is located at \( (x, y, z) = (0, 0, -0.05) \text{ m} \)?

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}
\]
\[
\vec{F} = q_2 \vec{E}
\]
\[
-e = -1.60 \times 10^{-19} \text{ C}
\]
\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2 \text{s}^2}{\text{kg m}^3}
\]

<table>
<thead>
<tr>
<th></th>
<th>( \frac{\text{kg m}}{s^2} \hat{y} )</th>
<th>( \frac{\text{kg m}}{s^2} \hat{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.7</td>
<td>-36</td>
</tr>
<tr>
<td>B</td>
<td>36</td>
<td>-36</td>
</tr>
<tr>
<td>C</td>
<td>-36</td>
<td>36</td>
</tr>
<tr>
<td>D</td>
<td>-36</td>
<td>36</td>
</tr>
<tr>
<td>E</td>
<td>36</td>
<td>36</td>
</tr>
</tbody>
</table>
Electric Potential

- Since electric force falls with $r^2$ just like gravity, the potential energy can just be written down by analogy.
- PE divided by $q_2$ is the electric potential which has units known as Volts.

\[
\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}
\]

force between two charges

\[
PE = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}
\]

potential energy

\[
\phi(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r}
\]

electric potential due to $q_1$
Some Electrical Terms

- **Charge**: a new fundamental quantity (C)
  - Charge Density: Charge per unit volume
- **Current**: Charge per second flowing through some plane \((A=\text{C/s})\).
- **Voltage**: Electric Potential Energy per unit charge.
- **Power** = \((\text{Voltage})(\text{Current})\)
Capacitor

- A capacitor can store charge
- Attraction between positive and negative charges helps keep the charge
- Energy is stored in the field
Parallel Plate Capacitor

• Electric field is nearly constant between plates.
• \( Q = VC \) defines Capacitance \( C \)

\[
C = \frac{A\varepsilon_0}{d}
\]
No Magnetic Charges

Divergence Of \( \mathbf{B} \)

\[
\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0
\]

\[
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0
\]

\[
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}
\]

\[
\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}
\]

\[
\vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}).
\]

B-field lines do not start or end.
Field lines are continuous.
B-field is proportional to the number of field lines per unit area.
Moving Charges give B-field Loops

\[ \vec{\nabla} \times \vec{B} = \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}, \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial z}, \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) \]

\[ \vec{\nabla} \cdot \vec{B} = 0 \]

\[ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \]

\[ \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

\[ \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \]

\[ \vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}). \]

**B-field lines loop around electric currents.**

**Right thumb along current, B-field along direction fingers curl.**

**Field lines are continuous.**
B Field of a Current in a Long Wire

Current \( I \) is the flow of electric charge. The Ampere is one Coulomb per second. The Ampere is defined to give a force of \( 2 \times 10^{-7} \) Newton per meter between parallel wires at a distance of one meter, each having one Ampere of current.

A current produces loops of magnetic field. Use the right hand to get the field direction.

\[
\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}
\]

Field of a long wire with current \( I \)

\[
\vec{F} = I (\vec{l} \times \vec{B})
\]

Force on wire of length \( l \) with current \( I \) in B-field

\[
\mu_0 = 4\pi \times 10^{-7} \text{ Tesla m/A}
\]
Question: B-field Lines

- Magnetic field lines:

\[
\begin{align*}
\nabla \cdot \vec{B} &= 0 \\
\n\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\n\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\
\n\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{J} \\
\vec{F} &= -e(\vec{E} + \vec{v} \times \vec{B}).
\end{align*}
\]

A) Start and end on charges.
B) Start and end on moving charges.
C) Form circles around charges.
D) Form circles around moving charges.
E) Start and end on accelerating charges.
Example: B-field of a long wire

- A very long wire has +1000 A of current passing through it along the x axis. What is the magnetic field at a point at \((x, y, z) = (0, 0, 0.1)\) meters?

\[
\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}
\]
\[
\vec{F} = I(\vec{I} \times \vec{B})
\]
\[
\mu_0 = 4\pi \times 10^{-7} \text{ Tesla m/A}
\]
\[
\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi} = \left(\frac{4\pi \times 10^{-7} \text{ Tesla m/A}}{2\pi (0.1 \text{ m})}\right) \hat{\phi}
\]
\[
\vec{B} = 2 \times 10^{-3} \hat{\phi} \text{ Tesla}
\]
\[
\vec{B} = -2 \times 10^{-3} \hat{y} \text{ Tesla \hspace{1cm} using right hand}
\]

This is larger than the field of the earth.
For large fields, electromagnets have many turns of wire in a coil.
Iron can multiply fields by a large number.
Problem: Field of a long wire

- A very long wire has $I = +5000\, \text{A}$ of current passing through it along the $y$ axis. What is the magnetic field at a point at $(x,y,z) = (0,0,0.02)\, \text{meters}$?

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \phi$$

$$\vec{F} = I(\vec{l} \times \vec{B})$$

$$\mu_0 = 4\pi \times 10^{-7}\, \text{Tesla m/A}$$

<table>
<thead>
<tr>
<th>Option</th>
<th>Magnetic Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>A)</td>
<td>$-0.004\hat{x}, \text{Tesla}$</td>
</tr>
<tr>
<td>B)</td>
<td>$0.05\hat{x}, \text{Tesla}$</td>
</tr>
<tr>
<td>C)</td>
<td>$1.0\hat{y}, \text{Tesla}$</td>
</tr>
<tr>
<td>D)</td>
<td>$0.05\hat{z}, \text{Tesla}$</td>
</tr>
<tr>
<td>E)</td>
<td>$-0.05\hat{z}, \text{Tesla}$</td>
</tr>
</tbody>
</table>
Problem: Field of a long wire

- A very long wire has $I=+5000$ A of current passing through it along the x axis. What is the magnetic field at a point at $(x,y,z)=(0,0,0.02)$ meters?

\[ \vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi} \]
\[ \vec{F} = I(\vec{l} \times \vec{B}) \]
\[ \mu_0 = 4\pi \times 10^{-7} \text{ Tesla m/A} \]

\[ A) \quad -0.05\hat{x} \text{ Tesla} \]
\[ B) \quad 0.05\hat{x} \text{ Tesla} \]
\[ C) \quad 0.05\hat{y} \text{ Tesla} \]
\[ D) \quad 0.05\hat{z} \text{ Tesla} \]
\[ E) \quad -0.05\hat{y} \text{ Tesla} \]
Simple Electromagnet

- **Solenoid**
- Nearly constant B-field inside.
- \( B = \mu_0 nI \)
- Field can be greatly enhanced by wrapping coil around iron.
EXB

• Force on a moving charge in a magnetic field
Magnet in Tube

- Permanent magnet induces eddy currents that try to keep the magnet from moving
- Lenz’s law
- Permanent magnets are caused by interesting Quantum Mechanical phenomena.
- Attractive and repulsive forces between magnets.
- Forces on currents.
- Can be used in motors and generators.
Swinging Plates in B Field

• Another illustration of Lenz’s law
  o EM always tries to oppose change
Generating Electric Power

Curl of \( \vec{E} \)

\[
\nabla \times \vec{E} = \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)
\]

Time varying B-field

\[
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0
\]

\[
\nabla \cdot \vec{B} = 0
\]

\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}
\]

\[
\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}
\]

\[
\vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}).
\]

E-field lines loop around changing B-fields. These lines do not start or end on charges. Field lines are continuous. We use this to generate electricity.
Motor and Generator

• A time varying magnetic field generates an electric field.
  o A time varying flow of B-field through a loop of wire can be used to generate electric power.
  o Use some source of energy to turn a generator and produce electric power.

• Similarly, we can use electric power to generate a magnetic force to make a motor.
Question: Generator

- We can generate electric power by:

\[
\begin{align*}
\nabla \cdot B & = 0 \\
\n\nabla \times \vec{E} + \frac{\partial B}{\partial t} & = 0 \\
\n\nabla \cdot \vec{E} & = \frac{\rho}{\varepsilon_0} \\
\n\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} & = \mu_0 \vec{J} \\
\vec{F} & = -e(\vec{E} + \vec{v} \times \vec{B}).
\end{align*}
\]

A) Turning a coil of wire in a B field.
B) Turning a coil of wire in a E field.
C) Passing a current through a coil of wire.
D) Both A and B
E) All of the above
End Quiz 4
B-field loops from varying E-field

\[ \nabla \times \vec{B} = \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}, \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial z}, \frac{\partial B_x}{\partial y} - \frac{\partial B_z}{\partial x} \right) \]

\[ \nabla \cdot \vec{B} = 0 \]

\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \]

\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \]

\[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \]

\[ \vec{F} = -e(\vec{E} + \vec{v} \times \vec{B}). \]

B-field lines loop around changing E-fields.
Field lines are continuous.
Effectively keeps flow of (displacement) current through capacitor.
Electromagnetic Waves

• Changing E field generates B field perpendicular.
• Changing B field generates E field perpendicular.
• Transverse vector wave
  o E, B, and direction of propagation are all perpendicular.
• Two possible polarizations.
Question: Light Wave

If a light wave is moving in the y direction, in which direction can the Electric field point?

A) x direction
B) y direction
C) z direction
D) A or C
E) Any of the above
Polarized Light

- Select light polarized in the x direction
- Rotate second polarizer to get maximum transmission or zero transmission
- Corn Syrup rotates polarization
But Light Wave is Composed of Quanta

- Diffraction is a property of waves.
- If we turn down the intensity of the light we detect single photons hitting the detector with \( E=hf \).
- They slowly fill up the diffraction pattern when the distribution is summed up over time.
  - Indicates that single photon interferes with itself.
  - Probability (amplitude) waves interfere
    → We will study this later.
Question: Diffraction

- The diffraction of light is evidence that light:

A) is composed of particles
B) is a wave
C) has two polarizations
D) is a transverse wave
E) comes from a laser

Answer: C
Radio Waves

• Static Charges cause static E-fields.
• Constant currents (of charge) cause static B-fields.
• Accelerating charges cause EM radiation.
  o Put oscillating signal onto antenna
  o Radiates waves with the frequency of the signal
  o E-field polarized in vertical direction.
• Use another antenna to pick up E-field in waves.
Maxwell’s Equations

Fields obey equations

\[
\begin{align*}
\nabla \cdot \vec{B} &= 0 \\
\n\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\n\nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\n\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{J} \\
\vec{F} &= -e(\vec{E} + \vec{v} \times \vec{B}).
\end{align*}
\]

Equations are fairly complex with 3 constants. But they are very practical for real problems (for slow humans).
Heaviside, Lorentz, Fitzgerald

Working on Electrodynamics of moving bodies

Oliver Heaviside (1850-1925) was a telegrapher, but deafness forced him to retire and devote himself to investigations of electricity. He became an eccentric recluse, befriended by FitzGerald and (by correspondence) by Hertz. In 1892 he introduced the operational calculus (Laplace transforms) to study transient currents in networks and theoretical aspects of problems in electrical transmission. In 1902, after wireless telegraphy proved effective over long distances, Heaviside theorized that a conducting layer of the atmosphere existed that allows radio waves to follow the Earth's curvature. He invented vector analysis and wrote Maxwell's equations as we know them today. He showed how EM fields transformed to new inertial frames.

Hendrik Antoon Lorentz (1853-1928), a professor of physics at the University of Leiden, sought to explain the origin of light by the oscillations of charged particles inside atoms. Under this assumption, a strong magnetic field would effect the wavelength. The observation of this effect by his pupil, Zeeman, won a Nobel prize for 1902 for the pair. However, the Lorentz theory could not explain the results of the Michelson-Morley experiment. Influenced by the proposal of Fitzgerald, Lorentz arrived at the formulas known as the Lorentz transformations to describe the relation of mass, length and time for a moving body. These equations form the basis for Einstein's special theory of relativity. Einstein read Lorentz's book

George Francis FitzGerald (1851-1901), a professor at Trinity College, Dublin, was the first to suggest that an oscillating electric current would produce radio waves, laying the basis for wireless telegraphy. In 1892 FitzGerald suggested that the results of the Michelson-Morley experiment could be explained by the contraction of a body along its its direction of motion.

Einstein
Field Equations in 4D

- $A_\nu$ is a 4-vector field
- $x_\nu$ is the coordinate 4-vector
- $j_\nu$ is the charge-current 4-vector
  - Source of EM field
- $c=1$
- Heaviside Lorentz units
- No constants
- Wave equation with source term.

Einstein notation: sum over repeated index

\[
\frac{\partial}{\partial x_\nu} \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) = \frac{j_\mu}{c} 
\]

One equation in 4-vector field with a 4-vector source (charge-current density) replaces Maxwell’s equations

Field tensor

\[
F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}
\]

\[
F_{\mu\nu} = \begin{pmatrix}
0 & B_z & -B_y & -iE_x \\
-B_z & 0 & B_x & -iE_y \\
B_y & -B_x & 0 & -iE_z \\
iE_x & iE_y & iE_z & 0
\end{pmatrix}
\]
### Question: EM in 4D

- The 4D equation below:

$$\frac{\partial}{\partial x_\nu} \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) = \frac{j_\mu}{c}$$

<table>
<thead>
<tr>
<th>Option</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A)</td>
<td>Has a source term $j_\mu$</td>
</tr>
<tr>
<td>B)</td>
<td>Is a 4-vector equation.</td>
</tr>
<tr>
<td>C)</td>
<td>Replaces all 4, 3D Maxwell equations.</td>
</tr>
<tr>
<td>D)</td>
<td>Both A and B</td>
</tr>
<tr>
<td>E)</td>
<td>All of the above.</td>
</tr>
</tbody>
</table>

The correct answer is **E). All of the above.**
And there was Light

- Visible light, IR, UV, microwaves, radio waves, x-rays, γ-rays are all Electromagnetic waves.
- Light and Electromagnetic Field are the same thing.
- Electric Field comes from charges.
- Magnetic Field comes from moving charges.
  - Really just Electric field transformed to moving coordinate system.
  - Symmetry of space-time.
- Vector Field
- EM field is also composed of photons

\[
\frac{\partial}{\partial x^v} F_{\mu v} = j_{\mu}
\]

\[
F_{\mu v} = \frac{\partial A_v}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_v}
\]
EM Field Does Amazing Things

- Changing magnetic field has an effect on charged particles which are outside the field.
- While the B field doesn’t touch the wire, the potential A-field does.
Field Equations in 4D

• EM equation is quite simple in 4D with the right units.
• Wave equation with source term.
• Probably the simplest vector field theory possible.
• A strange new symmetry called “gauge symmetry”.
• Einstein wanted to unify E&M with General Relativity.
  o He failed
  o But Kaluza did it and it may have some truth to it.

\[
\frac{\partial}{\partial x_\nu} \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) = \frac{j_\mu}{c}
\]

One equation in 4-vector field with a 4-vector source (charge-current density) replaces Maxwell’s equations

\[
F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}
\]

Field tensor

\[
F_{\mu\nu} = \begin{pmatrix}
0 & B_z & -B_y & -iE_x \\
-B_z & 0 & B_x & -iE_y \\
B_y & -B_x & 0 & -iE_z \\
iE_x & iE_y & iE_z & 0
\end{pmatrix}
\]
Question: 3D or 4D

\[
\frac{\partial}{\partial x_\nu} \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) = \frac{j_\mu}{c}
\]

• Which is false?

A) Maxwell’s equations in 4D are relatively simple.
B) It's easier to use the 3D Maxwell equations to solve real problems like capacitors and electromagnets.
C) The SI units chosen for Coulombs, Amperes and fields were not chosen well.
D) The simplicity of the 4D equations is further evidence that we live in 4D.
E) None of the above.
Kaluza-Klein Theory

- Kaluza (1919) postulated an invisible fifth dimension.
- Klein said it was invisible because it was curled up into a small circle.
- There is a circle at every point in 4D.
  - The circle is very small.
- There is a symmetry that we can be anywhere on the circle U(1).
- And there is a symmetry under rotations in 5D.
- E&M may just be GR in this invisible dimension.
- In modern theories there are more extra dimensions to accommodate the other fields.
  - There is no proof of extra dimensions yet.
Quantum Mechanics

Probability Amplitude Waves
for all particles

\[ \text{Probability}(\vec{r}, t) = |\psi(\vec{r}, t)|^2 \]
\( \psi \) is a "probability amplitude"
The Other Big Surprise of 1905

• Physics described mechanics, gravity, E&M fairly well.

• But there were many things that were not described.
  o Why all the elements and their properties?
  o Atomic emission energies.
  o Photoelectric effect (Einstein 1905).

• There was still a lot of physics to learn.
Photoelectric Effect

• Photons are particles of light with $E=hf$.
  o We need high frequency light to knock electrons out of metal plate.

• EM waves would not have this property.
  o Waves of sufficient intensity would knock electrons out independent of frequency.

\[ E_{\text{photon}} = h\nu \]
Problems with Classical Physics

• Too many states in the EM field.
  o All energy would be (rather instantly) radiated into EM field leaving everything at zero temperature.

• Atomic electrons should radiate and fall into nucleus.
  o Atoms would not exist.

• Atoms observed to emit light at quantized wavelengths.

• Light hitting metals could eject electrons from the metals as if $E=hf$.
  o $h$ is Plank’s constant (small number).
Question: Classical Physics

Which effect was understood in classical physics? (before quantum mechanics)

A) Interference of light waves
B) Emission of light at quantized frequency by atoms
C) Photoelectric effect
D) Size of atoms
E) None of the above.

Answer: C
Electromagnetic Waves?

- EM waves travel at speed of light
  - Maximum possible speed
- Waves can interfere as evidenced by diffraction.
- However light is always detected in quanta (particles) with $E=hf$.
  - Einstein also published photoelectric effect in 1905
    - Particles are called photons
- We can get diffraction patterns using one photon at a time!
- Each photon is described by a wave.
- Quantum Mechanics!

\[
\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-20} \frac{\text{kg} \text{m}^2}{\text{s}}
\]
Two Slit Diffraction

- Diffraction is a wave phenomenon.
- With 2 slit diffraction, we can get several maxima and minima.
Diffraction

- Turn down beam intensity so there is only one photon in the apparatus at a time.
- Always detect one photon.
- Probability distribution shows same diffraction pattern!
A-field Satisfies Wave Equation

- Solutions are waves
- E and B are just derivatives of A
- Energy in field proportional to squares of fields.
- Probability to find a photon goes like square of fields.
- Probability Amplitude Waves
  - Amplitudes can interfere
  - Probability gives diffraction pattern
  - Yet we detect quantized photons
  - With $E=hf$
Other Particles also

Probability Amplitude Waves

- Electrons, quarks, neutrinos…
- Waves of probability amplitude $\psi$
  - $P = |\psi|^2$
  - $\psi$ complex wave function
    - Complex number: $a + ib$
    - $i = \sqrt{-1}$
- Satisfy Schrödinger equation (wave)
  \[
  \frac{-\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t) + V(\vec{x}) \psi(\vec{x}, t) = i\hbar \frac{\partial \psi}{\partial t}
  \]
  \[
  H\psi = i\hbar \frac{\partial \psi}{\partial t}
  \]
Electron Diffraction

- Use crystal model too
- Electron diffraction with 2 slits is a very difficult experiment
- We can more easily see electron diffraction from a crystal with many atoms arranged in a regular array.
- Strong evidence that electrons are also waves.
Probability Amplitudes can Interfere

• Amplitudes for particles to arrive at one place from two different sources can interfere.
  \[ \psi = \psi_1 + \psi_2 \]
  \[ \text{prob.} = |\psi_1 + \psi_2|^2 \]

• Describes diffraction and various other experiments well.

• If one observes which slit the particle went through, the diffraction pattern is spoiled.
Question: Particle Wave Duality

How do we explain the wave behavior of electrons and photons at the same time as the fact that they are always detected as quantized particles.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A) Sometimes particles are waves and sometimes not.</td>
<td></td>
</tr>
<tr>
<td>B) Probability to find a particle is proportional to square of a wave.</td>
<td></td>
</tr>
<tr>
<td>C) Probability to find a particle proportional to wave.</td>
<td></td>
</tr>
<tr>
<td>D) Particles are always waves.</td>
<td></td>
</tr>
<tr>
<td>E) Waves are always made of particles.</td>
<td></td>
</tr>
</tbody>
</table>

Answer: C
Photons

• Photons have zero mass
• Energy is related to frequency
  o $E=hf$ according to Plank and Einstein
  o Frequency is related to wavelength: $\lambda f = c$
• $E=pc$ since mass is zero
• $\lambda = h/p$ relation between momentum and wavelength.

\[
E = hf
\]
\[
\lambda = \frac{h}{p}
\]
\[
h = 2\pi\hbar
\]
Wavefunctions for Electrons…

- Wavefunctions for matter particles can be pure waves with one frequency and one wavelength.

\[ \psi_\vec{p} = e^{i(\vec{p} \cdot \vec{x} - Et)/\hbar} \]

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

\[ i = \sqrt{-1} \]

\[ P = |\psi|^2 = 1 \]

- Momentum related to wavelength
- Energy related to frequency
- Particle’s probability is completely spread out over all space.
Localize Electrons with Wavepackets

- Wavepackets can be localized by combining waves with different wavelengths.
  - Principle of superposition.
- To localize well, we need a wide spectrum of wavelengths.
- This gives rise to an uncertainty principle which is a basic property of waves.
  - Math too complicated to show here.
  - But result is simple.

\[ \Delta p \Delta x \geq \frac{\hbar}{2} \]
Fast Fourier Transform

• We can describe a pulse either as \( g(t) \) or as \( h(f) \).
Wave Nature Implies Uncertainty Principle

- **Heisenberg uncertainty principle**
  \[ \Delta p \Delta x \geq \frac{\hbar}{2} \]

- **We cannot know a particle’s momentum and position at the same time.**

- **This is a basic property of waves.**
  \[ \Delta k \Delta x \geq \frac{1}{2} \]
  \[ p = \hbar k \]
Question: Uncertainty

If we try to localize an electron’s position very well, what other property must become very uncertain.

A) Its charge
B) Its mass
C) Its momentum
D) All of the above
E) None of the above

Answer: C
Electron Volts (Unit of Energy)

- One electron Volt is the energy a particle with charge $e$ gets by going through one Volt of Electric potential difference.
  - $eV$ is a convenient atomic physics unit.
  - It is quite convenient for accelerators.
- MeV (million electron Volts) is a good unit for nuclei and accelerators.
- $eV$, keV, MeV, GeV, TeV
  - 1, 1000, 1000000, 1000000000, 1000000000000
Estimate Hydrogen Energy using Uncertainty Principle

\[ E = \frac{1}{2} mv^2 - \frac{e^2}{4\pi\varepsilon_0 r} \]

Energy=KE+PE

\[ E = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r} \]

use momentum \( mv \)

\[ \Delta p \Delta r \approx \hbar \]

uncertainty

\[ p \approx \Delta p \]

min. \( p \) is \( \Delta p \)

\[ r \approx \Delta r \]

min. \( r \) is \( \Delta r \)

\[ pr \approx \hbar \]

\[ E = \frac{p^2}{2m} - \frac{e^2 p}{4\pi\varepsilon_0 \hbar} \]

replace \( r \)

\[ \frac{dE}{dp} = \frac{p}{m} - \frac{e^2}{4\pi\varepsilon_0 \hbar} = 0 \] \quad \text{min } E

\[ p = \frac{me^2}{4\pi\varepsilon_0 \hbar} \]

solve

\[ E = \frac{p^2}{2m} - \frac{e^2 p}{4\pi\varepsilon_0 \hbar} \]

\[ E = \frac{me^4}{2(4\pi\varepsilon_0 \hbar)^2} - \frac{me^4}{(4\pi\varepsilon_0 \hbar)^2} \]

\[ \alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137} \] \quad \text{EM constant}

\[ E = -\frac{me^4}{2(4\pi\varepsilon_0 \hbar)^2} = -\frac{\alpha^2 mc^2}{2} \]

\( E=-13.6 \text{ eV}, \) the right answer
Atoms

• Atomic energies and sizes are set by uncertainty principle.
  o $E = -13.6$ eV Hydrogen ground state energy
  o $a_0 = 0.53 \times 10^{-10}$ m Bohr radius

• Comes from wavefunction for electron.
  o Can’t make both $p$ and $r$ small
Another Uncertainty Principle

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \]

- Can violate energy conservation for short periods of time
  - Quantum tunneling
  - Virtual particles
- Stable states have definite energies
  - \( \Delta E = 0 \)
  - \( \Delta t = \text{infinity} \)
  - Eigenvalue Solution to wave equation
    \[ H \psi = E \psi \]
  - If E is measured, must get one of the eigenvalues
Schrödinger Equation

• The Schrödinger Equation is a wave equation.
• Equation for stationary states replaces time derivative by energy.

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t) + V(\vec{x})\psi(\vec{x}, t) = E\psi(\vec{x}, t) \]
\[ H\psi = E\psi \]

• Solutions exist only for some energies.
  o Quantized energies.
  o Only these energies can be measured.
Hydrogen Energies

• A solution to the Schrödinger equation gives the Hydrogen energies in terms of quantum numbers.

\[ \Psi_{n\ell m} \] Definite energy wavefunction
\[ n = 1, 2, 3, \ldots \] Principle quantum number for Hydrogen
\[ \ell = 0, 1, \ldots, n - 1 \] integer: total angular momentum quantum number
\[ -\ell \leq m \leq \ell \] integer: z component of angular momentum is \( m\hbar \)

\[ E_{n\ell m} = -\frac{\alpha^2 mc^2}{2n^2} \] energies only depend on \( n \) (+ small corrections)
\[ E_{n\ell m} = -\frac{13.6}{n^2} \text{ eV} \]
\[ \alpha = \frac{1}{137} \] dimensionless
Energy “Eigenstates”

- Atoms have definite energy states.
- Ground state lives forever.
- Excited states decay
  - Have some small E width

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \]

Energy levels of the hydrogen atom with some of the transitions between them that give rise to the spectral lines indicated.
We can calculate the quantized energies allowed for atoms by:

A) Using the uncertainty principle.
B) Solving the Schrödinger equation.
C) They can only be determined by experiment.
D) Solving Maxwell’s equations.
E) Using the science of Chemistry.
A diffraction grating is a large array of evenly spaced scratches,
Light of one wavelength from each scratch will interfere constructively at the correct angle.
We can see the energies of light from atomic decays using a diffraction grating.

**Emission spectrum**

**Absorption spectrum**

![Emission spectrum](image)

![Absorption spectrum](image)
Question: Energy Spectra

- We can use diffraction gratings to

  A) Measure emission energies of atoms
  B) Measure absorption energies of atoms
  C) Separate light of different wavelengths
  D) A and B
  E) All of the above
Electrons are Spin 1/2

- Internal angular momentum
- Two spin states (up or down) for any axis
  - Spin is a new internal coordinate for electrons
- Identical particles
  - Fermions for spin 1/2
  - No two particles in the same state
  - Atomic levels fill up
  - Matter particles

Electron Spin

- Uhlenbeck & Goudsmit proposed electron had “intrinsic” angular momentum.
- Spin UP means spin parallel to B field, and vice-versa.
Other Phenomena of Quantum Mechanics

- Atomic energy levels, nuclei...
- Atomic decay rates
- Permanent magnets
- Metallic conductors
- Energy bands in crystals
- Semiconductors
- Tunneling
- Superconductivity
- Lasers
Photons are Spin 1 (vector)

- Internal angular momentum
  - The two polarizations
- Identical particles
  - Bosons for integer spin
  - Can be emitted or absorbed by charged particles
  - Many particles in the same state
  - Energy particles
  - EM Force carriers
Question: Identical Particles

• Which is true about the states of identical particles?

A) No two identical particles in the same state
B) No two identical Fermions in the same state
C) No two identical Bosons in the same state
D) No two particles in the same state
E) None of the above
Gauge Symmetry

- A strange symmetry of wave functions and EM field
- Extremely important in our theories of EM, and other fields
- We can change the phase of the wavefunction by a different amount at every point in space-time.
  - We must make a related change in the EM potential $A_\mu$ at every point.
- Gauge symmetry keeps photon massless.
- Is this really some space-time symmetry in extra dimensions?
  - A curled up extra dimension can give EM gauge theory
Relativity and Quantum Mechanics

• It was hard to merge relativity and Quantum Mechanics.
• Physicists were puzzled by the energy relation in relativity.
• When they solved the wave equations they got solutions with both positive and negative energy.
  - And they could show they had to keep the negative energy solutions.
• Dirac made a big effort to find an equation that was linear in the energy (operator).
  - He found a matrix equation that represents electrons well
  - But it still had negative energy solutions.

\[ E = \pm \sqrt{(pc)^2 + (mc^2)^2} \]
Relativistic Quantum Mechanics

• The wave equations of E&M already represent the relativistic equation for photons well.
  o They just need to be quantized
• Dirac’s equation is good for electrons.
• But it has negative energy solutions.

\[
\frac{\partial}{\partial x_v} F_{\mu v} = j_\mu
\]

\[
F_{\mu v} = \frac{\partial A_v}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_v}
\]

\[
(\gamma^\mu \frac{\partial}{\partial x_\mu} + m)\Psi = 0
\]
Dirac Equation

\[ (\gamma^\mu \frac{\partial}{\partial x_\mu} + m)\Psi = 0 \]

- \( \gamma \) is a 4X4 matrix
- \( \psi \) is a 4 component spinor
  - 2 spin states
    - Times 2 energy solutions, + and -
- Automatically includes spin.
- Negative energy states reinterpreted.
Antiparticles

• Negative energy could be reinterpreted as positive energy moving backward in time.

\[ \psi_\vec{p} = e^{i(\vec{p} \cdot \vec{x} - Et)/\hbar} \]

  - Time reversal symmetry in solutions

• Particles that move backward in time, appear to us as having the opposite charge...
  - Gives us pair production of \( e^+e^- \)
  - Pair annihilation

• Dirac predicted Antimatter.
Quantum Electrodynamics

\[
\frac{\partial}{\partial x_v} F_{\mu\nu} = j_\mu \quad F_{\mu\nu} = \frac{\partial A_v}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_v} \quad (\gamma^\mu \frac{\partial}{\partial x_\mu} + m)\Psi = 0
\]

- **QED** is quantum field theory of electrons and photons.
- Written in terms of electron field \(\psi\) and photon field \(A_\mu\).
- Fields are quantized.
  - Able to create or annihilate photons with \(E=hf\).
  - Able to create or annihilate electron positron pairs.
- **Gauge (phase) symmetry transformation**
  - Only gauge theory we could have with 1 photon!
- **Extremely good calculational accuracy**
  - 15 digits
Relativistic Quantum Field Theory

• Dirac Equation: Relativistic QM for electrons
  o Matrix ($\gamma$) eq. Includes Spin, Antiparticles
  o Negative E solutions understood

• Quantum Electrodynamics
  o Field theory for electrons and photons
  o Rules of QFT developed and tested
    → Lamb Shift
    → Vacuum Polarization
  o Example of a “Gauge Theory”

• Weak and Strong Interactions understood as gauge theories like QED around 1970

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\Psi = 0$$

Feynman Diagrams