# Physics 2D Lecture Slides
# Mar 5

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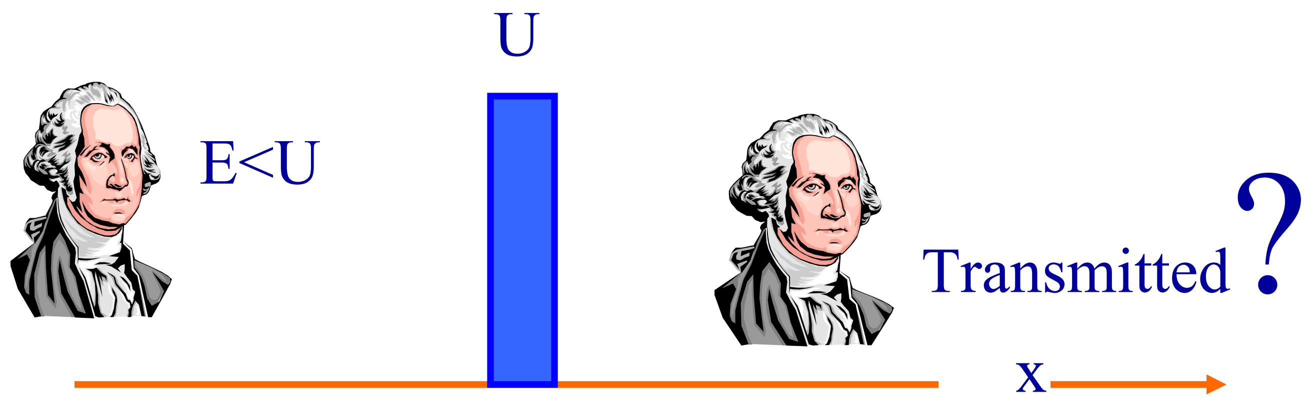
# Potential Barrier



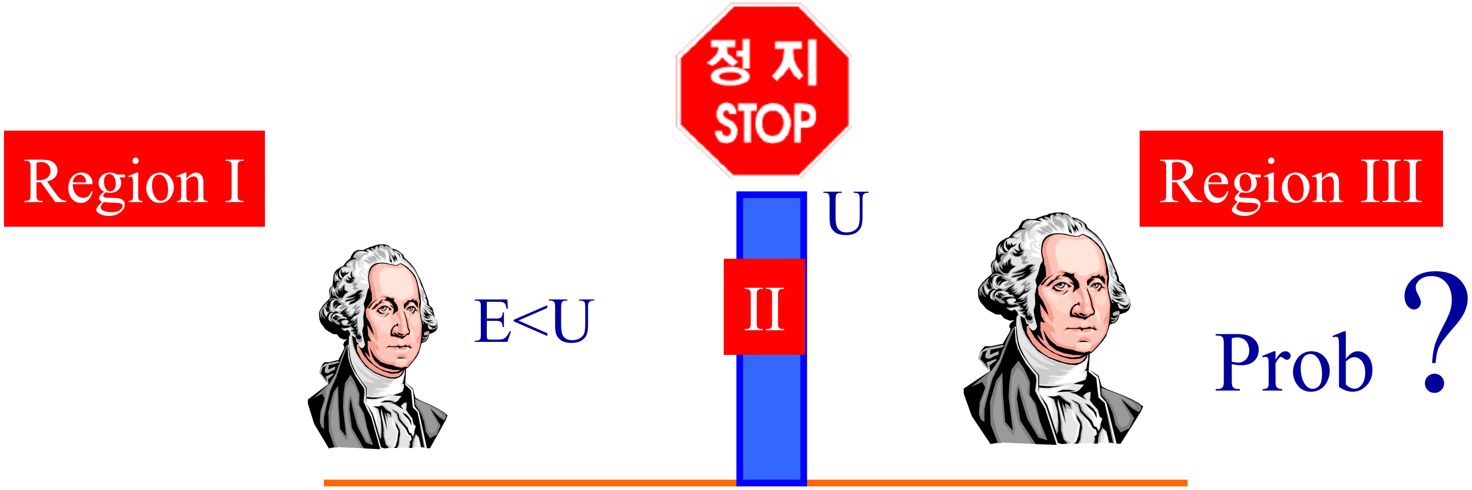

| Description of Potential | | |
|---|---|---|
| U = 0 | x < 0 | (Region I ) |
| U = U | 0 < x < L | (Region II) |
| U = 0 | x > L | (Region III) |

Consider George as a “free Particle/Wave” with Energy E incident from Left

Free particle are under no Force; have wavefunctions like

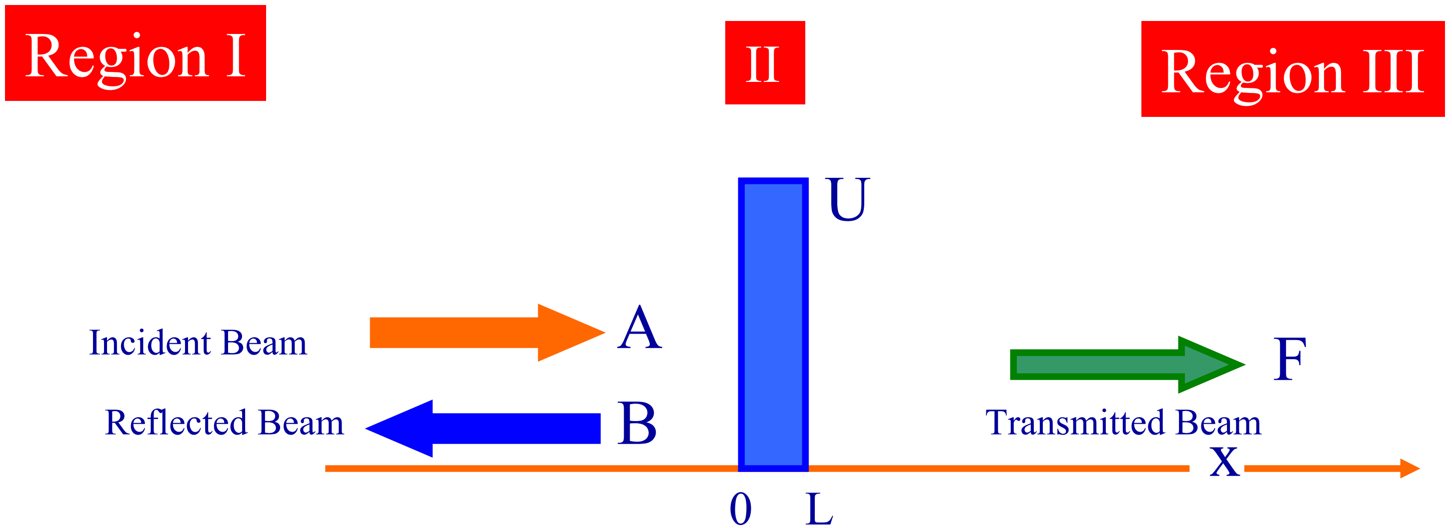
$$\Psi = A\, e^{i(kx-wt)} \text{ or } B\, e^{i(-kx-wt)}$$

# Tunneling Through A Potential Barrier




- Classical & Quantum Pictures compared: When E>U & when E<U
- Classically , an particle or a beam of particles incident from left encounters barrier:
    - when E > U → Particle just goes over the barrier (gets transmitted )
    - When E<U → particle is stuck in region I, gets entirely reflected, no transmission (T)
- What happens in a Quantum Mechanical barrier ? No region is inaccessible for particle since the potential is (sometimes small) but finite

# Beam Of Particles With E < U Incident On Barrier From Left




Description Of WaveFunctions in Various regions: Simple Ones first

In Region I : $\Psi_{\mathrm{I}}(x,t) = Ae^{i(kx-\omega t)} + Be^{i(-kx-\omega t)}$ = incident + reflected Waves

$$\text{with } E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

*define* Reflection Coefficient : $R = \frac{|B|^2}{|A|^2}$ = *frac* of incident wave intensity reflected back

In Region III: $\Psi_{\mathrm{III}}(x,t) = Fe^{i(kx-\omega t)} + Ge^{i(-kx-\omega t)}$ = *transmitted*

*Note* : $Ge^{i(-kx-\omega t)}$ corresponds to wave incident from right !

This piece does not exist in the scattering picture we are thinking of now (G=0)

So $\Psi_{\mathrm{III}}(x,t) = Fe^{i(kx-\omega t)}$ represents transmitted beam. Define $T = \frac{|F|^2}{|A|^2}$

*Unitarity* Condition $\Rightarrow$ R + T= 1 (particle is either reflected or transmitted)

# Wave Function Across The Potential Barrier

In Region II of Potential U

TISE: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U\psi(x)=E\psi(x)$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2}=\frac{2m}{\hbar^2}(U-E)\psi(x)$$

$$=\alpha^2\psi(x)$$

with $\alpha=\frac{\sqrt{2m(U-E)}}{\hbar}$ ; U>E $\Rightarrow \alpha^2>0$

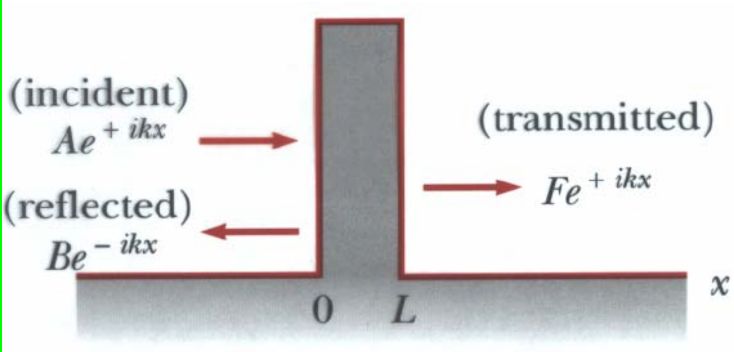
Solutions are of form $\psi(x)\propto e^{\pm\alpha x}$

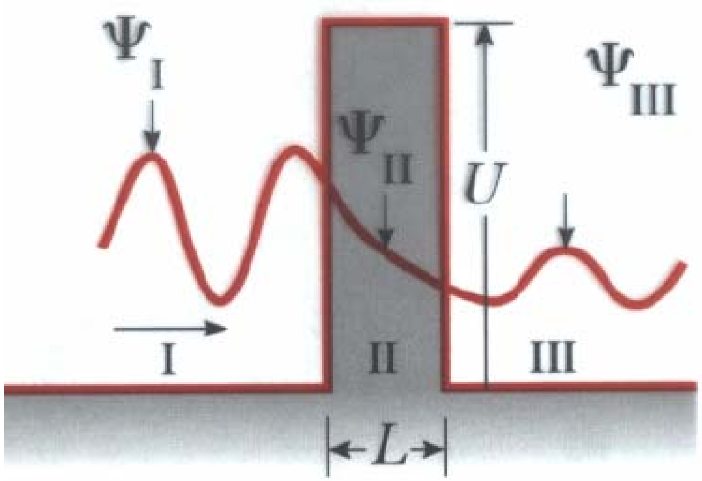
$$\Psi_{II}(x,t)=Ce^{+\alpha x-i\omega t}+De^{-\alpha x-i\omega t} \quad 0<x<L$$

To determine C & D $\Rightarrow$ apply matching cond.

$\Psi_{II}(x,t)=continuous$ across barrier (x=0,L)

$\frac{d\Psi_{II}(x,t)}{dx}=continuous$ across barrier (x=0,L)



# Continuity Conditions Across Barrier

At x = 0 , continuity of $\psi(x) \Rightarrow$

$$A+B=C+D \quad (1)$$

At x = 0 , continuity of $\dfrac{d\psi(x)}{dx} \Rightarrow$

$$ikA - ikB = \alpha C - \alpha D \quad (2)$$

Similarly at x=L continuity of $\psi(x) \Rightarrow$

$$Ce^{-\alpha L} + De^{+\alpha L} = Fe^{ikL} \quad (3)$$

at x=L, continuity of $\dfrac{d\psi(x)}{dx} \Rightarrow$

$$-(\alpha C)e^{-\alpha L} + (\alpha D)e^{\alpha L} = ikFe^{ikL} \quad (4)$$

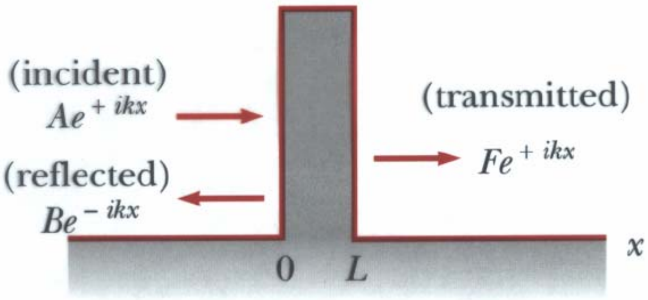
Four equations & four unknowns

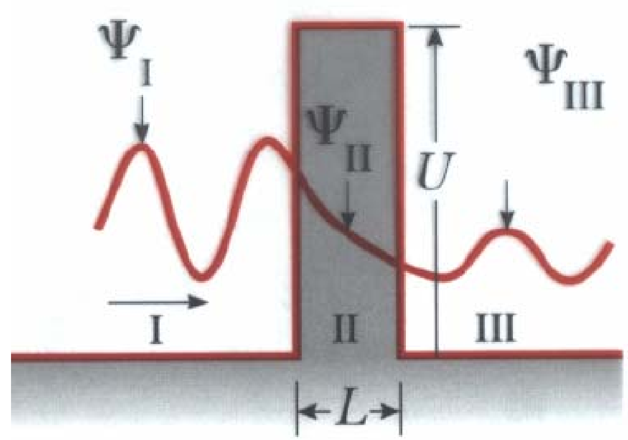
Cant determine A,B,C,D but if you

Divide thruout by A in all 4 equations :

$\Rightarrow$ ratio of amplitudes $\rightarrow$ relations for R & T

That's what we need any way



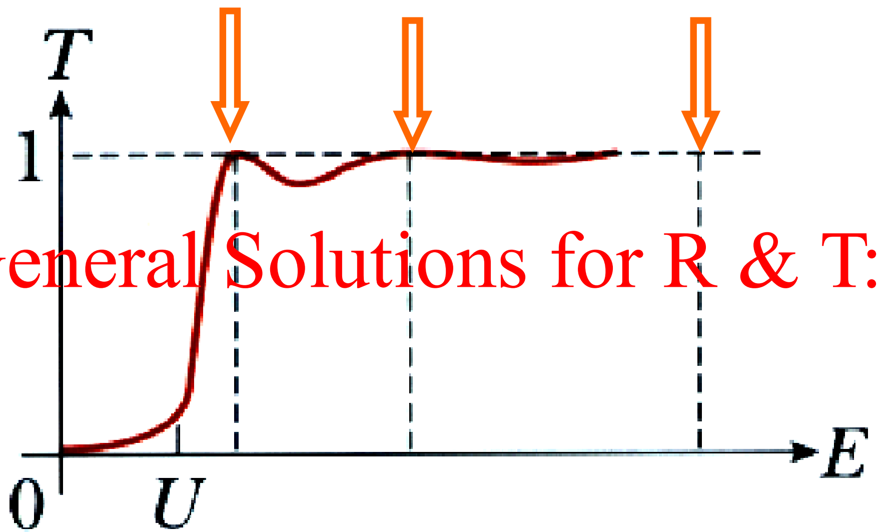
# Potential Barrier when E < U

Expression for Transmission Coeff T=T(E) :

Depends on barrier Height U, barrier Width L and particle Energy E

$$T(E) = \left[1 + \frac{1}{4}\left(\frac{U^2}{E(U-E)}\right)\sinh^2(\alpha L)\right]^{-1}; \quad \alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$$

and R(E)=1- T(E).........what's not transmitted is reflected




General Solutions for R & T:

Above equation holds only for E < U

For E>U, α=imaginary#

Sinh(αL) becomes oscillatory

This leads to an Oscillatory T(E) and

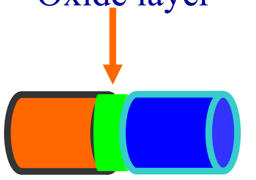
Transmission resonances occur where

For some specific energy ONLY, T(E) =1

At other values of E, some particles are reflected back ..even though E>U !!

That's the Wave nature of the Quantum particle

# Ceparated in Coppertino



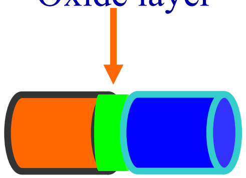

Solved Example 6.1 (...that I made such a big deal about yesterday)

Q: 2 Cu wires are seperated by insulating Oxide layer. Modeling the Oxide layer as a square barrier of height U=10.0eV, estimate the transmission coeff for an incident beam of electrons of E=7.0 eV when the layer thickness is (a) 5.0 nm (b) 1.0nm

Q: If a 1.0 mA current in one of the intwined wires is incident on Oxide layer, how much of this current passes thru the Oxide layer on to the adjacent wire if the layer thickness is 1.0nm? What becomes of the remaining current?

$$\mathrm{T(E)} = \left[1+\frac{1}{4}\left(\frac{U^2}{E(U-E)}\right)\sinh^2(\alpha L)\right]^{-1}$$

$$\alpha=\frac{\sqrt{2\mathrm{m(U\text{-}E)}}}{\hbar},\quad k=\frac{\sqrt{2\mathrm{mE}}}{\hbar}$$




$$T(E) = \left[1+\frac{1}{4}\left(\frac{U^2}{E(U-E)}\right)\sinh^2(\alpha L)\right]^{-1}$$

Use $\hbar$=1.973 keV.Å/c , $m_e$ = 511 keV/c$^2$

$$\Rightarrow \alpha = \frac{\sqrt{2m_e(U-E)}}{\hbar} = \frac{\sqrt{2\times 511 kev/c^2(3.0\times10^{-3} keV)}}{1.973 \text{ keV.Å/c}} = 0.8875\text{Å}^{-1}$$

Substitute in expression for T=T(E)

$$T= \left[1+\frac{1}{4}\left(\frac{10^2}{7(10-7)}\right)\sinh^2(0.8875\text{Å}^{-1})(50\text{Å})\right]^{-1} = 0.963\times10^{-38}(small)!!$$

However, for L=10Å; $T=0.657\times10^{-7}$

Reducing barrier width by $\times 5$ leads to Trans. Coeff enhancement by 31 orders of magnitude !!!

1 mA current $=I=\frac{Q=Nq_e}{t} \Rightarrow N=6.25\times10^{15}$ *electrons*

$N_T$=# of electrons that escape to the adjacent wire (past oxide layer)

$N_T = N.T = (6.25\times10^{15}\ electrons)\times \boxed{T}$;

For L=10 Å, $T=0.657\times10^{-7} \Rightarrow N_T = 4.11\times10 \Rightarrow \boxed{I_T = 65.7\,pA}$!!

Current Measured on the first wire is sum of incident+reflected currents and current measured on "adjacent" wire is the $I_T$

Oxide thickness makes all the difference !
That's why from time-to-time one needs to
Scrape off the green stuff off the naked wires

# A Special Case That is Instructive & Useful: U>>E

Given the 4 equations from Continuity Conditions: Solve for $\frac{\mathrm{A}}{\mathrm{F}}$

$$\frac{\mathrm{A}}{\mathrm{F}}=\left[\frac{1}{2}+\frac{i}{4}\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)\right]e^{(ik+\alpha)L}+\left[\frac{1}{2}-\frac{i}{4}\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)\right]e^{(ik-\alpha)L}$$

Remember $\alpha=\frac{\sqrt{2\mathrm{m}(\mathrm{U}-\mathrm{E})}}{\hbar},\ k=\frac{\sqrt{2\mathrm{mE}}}{\hbar},$

when U>>E, $\boxed{\alpha\gg k\ \&\ \frac{\alpha}{k}\gg\frac{k}{\alpha}}$ So $\frac{\alpha}{k}-\frac{k}{\alpha}\approx\frac{\alpha}{k}$; For large Barrier L, $\alpha L\gg 1$

$$\frac{\mathrm{A}}{\mathrm{F}}=\left[\frac{1}{2}+\frac{i}{4}\left(\frac{\alpha}{k}\right)\right]e^{(ik+\alpha)L};\left(\frac{\mathrm{A}}{\mathrm{F}}\right)^{*}=\left[\frac{1}{2}-\frac{i}{4}\left(\frac{\alpha}{k}\right)\right]e^{(-ik+\alpha)L}$$

$$\mathrm{T}^{-1}=\left(\frac{\mathrm{A}}{\mathrm{F}}\right)^{*}\frac{\mathrm{A}}{\mathrm{F}}=\left[\frac{1}{4}+\frac{1}{16}\left(\frac{\alpha^{2}}{k^{2}}\right)\right]e^{(2\alpha)L}=\frac{1}{T(E)};$$ now invert & consolidate

$$\mathrm{T}=\left|\frac{\mathrm{F}^{*}\mathrm{F}}{\mathrm{A}^{*}\mathrm{A}}\right|=\left(\frac{16}{4+\left(\frac{\alpha}{k}\right)^{2}}\right)e^{-2\alpha L}$$ ; now watch the variables employed

# A Special Case That is Instructive & Useful: U>>E

$$\alpha = \frac{\sqrt{2m(U-E)}}{\hbar},\ k = \frac{\sqrt{2mE}}{\hbar} = \frac{p}{\hbar} = \frac{2\pi}{\lambda}$$

$$\left(\frac{\alpha}{k}\right)^2 = \frac{2m(U-E)/\hbar^2}{2mE/\hbar^2} = \frac{U}{E} - 1 \simeq \frac{U}{E}$$

$$\Rightarrow \left(\frac{16}{4+\left(\frac{\alpha}{k}\right)^2}\right) = \left(\frac{16}{4+\left(\frac{U}{E}-1\right)}\right) \rightarrow \text{varies slowly compared with } e^{-2\alpha L}\ \textit{term}$$

keeing in mind only the Order of magnitude, I suggest

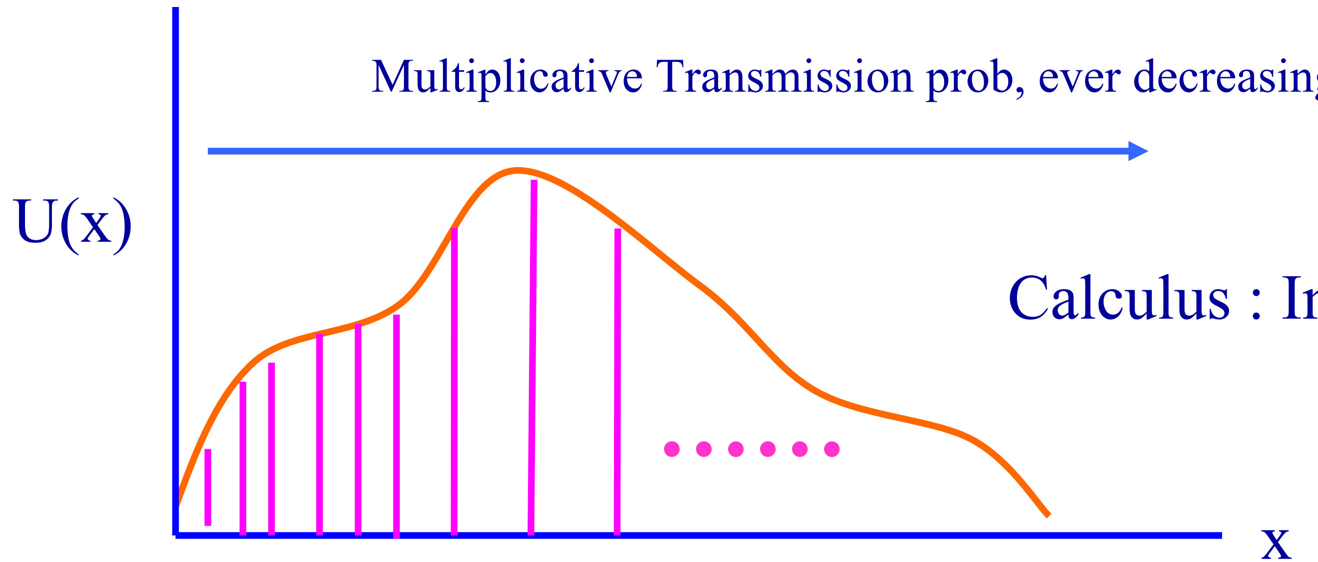
$$\left(\frac{16}{4+\left(\frac{U}{E}-1\right)}\right) \approx 1;\ \text{back to } T = \left(\frac{16}{4+\left(\frac{\alpha}{k}\right)^2}\right) e^{-2\alpha L} \quad \text{substituting}$$

So approximately $\boxed{T \approx e^{-2\alpha L}}$ Transmission Prob is fn of U,E,L

Why subject you to this TORTURE? → Estimate T for complicated Potentials

See next (example of Cu oxide layer, radioactivity, blackhole blowup etc)

# A Complicated Potential Barrier Can Be Broken Down



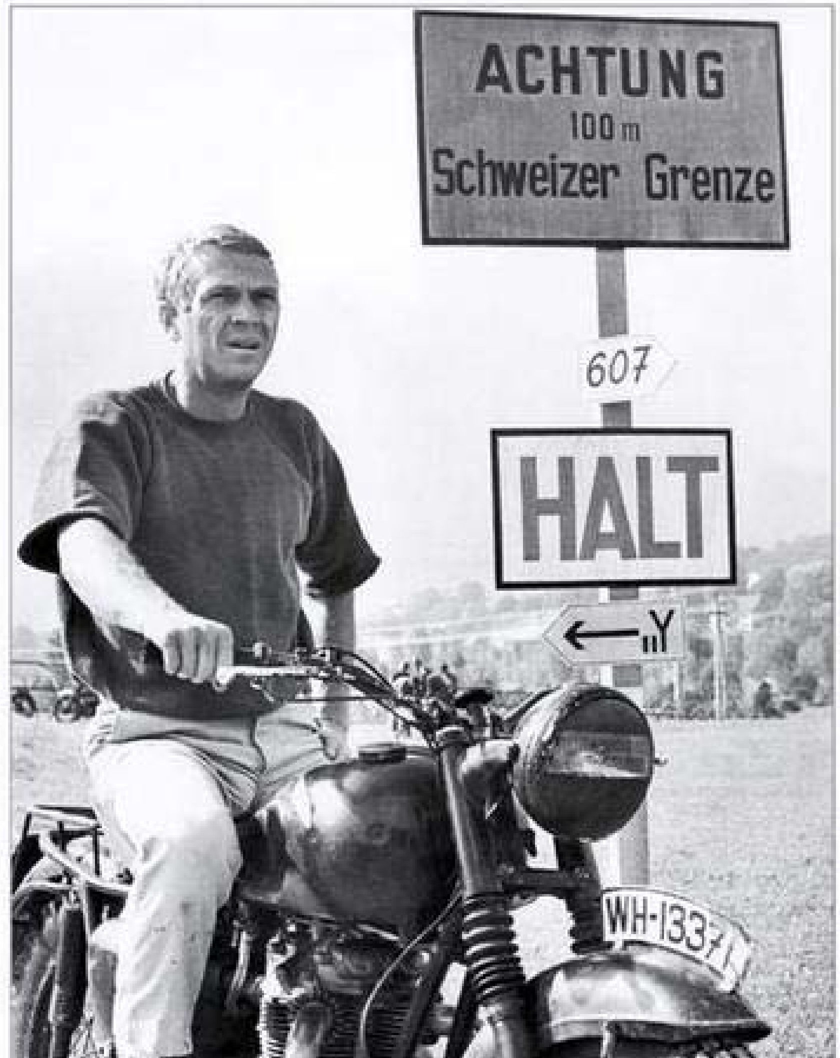

Can be broken down into a sum of successive Rectangular potential barriers for which we learnt to find the Transmission probability $T_i$

The Transmitted beam intensity thru one small barrier becomes incident beam intensity for the following one

So on & so forth …till the particle escapes with final Transmission prob T

$$T = \int T_i \, dx = e^{-2\left[\frac{\sqrt{2m}}{\hbar}\int\sqrt{U(x)-E}\,dx\right]}$$

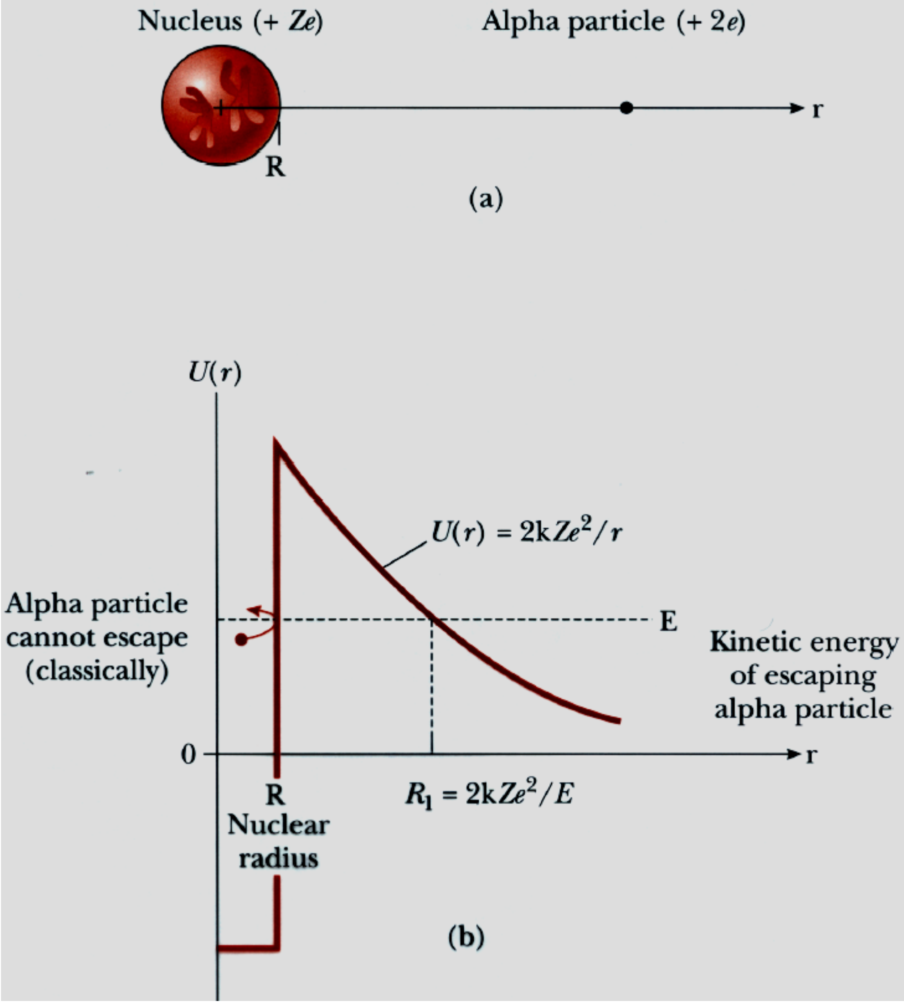
# The Great Escape !      My Favorite Movie


Story involves an Allied plan for a massive breakout from a Nazi P.O.W. camp, during World War Two. The Nazis had created a high-security, escape -proof prisoner of war camp for those annoying detainees who have attempted escape from their other prison of war camps.

These prisoners are not discouraged at all, as they plan a huge escape of 100 men.

# Radioactivity: The $\alpha$-particle & Steve McQueen Compared




- In a Nucleus such as Ra, Uranium etc $\alpha$ particle rattles around parent nucleus, "hitting"the nuclear walls with a very high frequency (probing the "fence"), if the Transmission prob T>0, then eventually particle escapes
- Within nucleus, $\alpha$ particle is virtually free but is trapped by the Strong nuclear force (see quiz), once outside nucleus, the particle "sees" only the columbic force (nuclear force too faint outside)
- Nuclear radius $R = 10^{-14}$ m, $E_\alpha = 9$MeV
- Coulomb barrier $U(r) = kq_1q_2/r$
  - At r=R, U(R) ≈ 30 MeV barrier
- $\alpha$-particle, due to QM, tunnels thru
- It's the sensitivity of T on $E_\alpha$ that accounts for the wide range in half-lives of radioactive nuclei
- See Eq.6.13 & DO Example 6.7

$$T = \int T_i \, dx = e^{-2\left[\frac{\sqrt{2m_\alpha}}{\hbar}\int\sqrt{\frac{2ke^2Z}{r} - E_\alpha}\,dx\right]}$$