Modern Physics

Second Edition

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Relativity

Chapter Outline

At the end of the 19th century, scientists believed that they had learned most of what there was to know about physics. Newton's laws of motion and his universal theory of gravitation, Maxwell's theoretical work in unifying electricity and magnetism, and the laws of thermodynamics and kinetic theory were highly successful in explaining a wide variety of phenomena.

However, at the turn of the 20th century, a major revolution shook the world of physics. In 1900 Planck provided the basic ideas that led to the formulation of the quantum theory, and in 1905 Einstein formulated his brilliant special theory of relativity. The excitement of the times is captured in Einstein's own words: "It was a marvelous time to be alive." Both ideas were to have a profound effect on our understanding of nature. Within a few decades, these theories inspired new developments and theories in the fields of atomic physics, nuclear physics, and condensed matter physics.

Although modern physics has been developed during this century and has led to a multitude of important technological achievements, the story is still incomplete. Discoveries will continue to evolve during our lifetime, many of which will deepen or refine our understanding of nature and the world around us. It is still a "marvelous time to be alive."

1.1 SPECIAL RELATIVITY

Light waves and other forms of electromagnetic radiation travel through free space at the speed $c = 3.00 \times 10^8$ m/s. As we shall see in this chapter, the speed of light is an upper limit for the speeds of particles and mechanical waves.

Most of our everyday experiences and observations deal with objects that move at speeds much less than that of light. Newtonian mechanics and early ideas on space and time were formulated to describe the motion of such objects, and this formalism is very successful for describing a wide range of phenomena. Although Newtonian mechanics works very well at low speeds, it fails when applied to particles whose speeds approach that of light. Experimentally, one can test the predictions of Newtonian theory at high speeds by accelerating an electron through a large electric potential difference. For example, it is possible to accelerate an electron to a speed of $0.99c$ by using a potential difference of several million volts. According to Newtonian mechanics, if the potential difference (as well as the corresponding energy) is increased by a factor of 4, then the speed of the electron should be doubled to $1.98c$. However, experiments show that the speed of the electron—as well as the speeds of all other particles in the universe—always remains less than the speed of light, regardless of the size of the accelerating voltage. Because it places no upper limit on the speed that a particle can attain, Newtonian mechanics is contrary to modern experimental results and is therefore clearly a limited theory.

In 1905, at the age of only 26, Albert Einstein published his *special theory of relativity*. Regarding the theory, Einstein wrote,

> The relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties, using only a few very convincing assumptions. . . .[1]

Although Einstein made many important contributions to science, the theory of relativity alone represents one of the greatest intellectual achievements of the 20th century. With this theory, one can correctly predict experimental observations over the range of speeds from $v = 0$ to speeds approaching the speed of light. Newtonian mechanics, which was accepted for over 200 years, is in fact a limiting case of Einstein's special theory of relativity. This chapter gives an introduction to the special theory of relativity which deals with the analysis of physical events from coordinate systems moving uniformly with respect to one another. A discussion of general relativity that describes events for coordinate systems undergoing general or accelerated motion is presented in an interesting essay written by Clifford Will at the end of the chapter.

In this chapter we show that the special theory of relativity is based on two basic postulates:

1. The laws of physics are the same in all reference systems which move uniformly with respect to one another. That is, basic laws such as $\Sigma \boldsymbol{F} = d\mathbf{p}/dt$ have the same mathematical form for all observers moving at constant velocity with respect to one another.

[1] A. Einstein and L. Infeld, *The Evolution of Physics*, New York, Simon and Schuster, 1961.

2. The speed of light in vacuum is always measured to be 3×10^8 m/s, and the measured value is independent of the motion of the observer or of the motion of the source of light. That is, the speed of light is the same for all observers moving at constant velocities.

Special relativity covers phenomena such as the slowing down of clocks and the contraction of lengths in moving reference frames as measured by an observer in another reference frame. We also discuss the relativistic forms of momentum and energy and some consequences of the famous mass-energy equivalence formula, $E = mc^2$.

Although it is well known that relativity plays an essential role in theoretical physics, it also has practical applications, including the design of accelerators and other devices that use high-speed particles. Note that these devices simply will not work if designed according to Newtonian mechanics! We shall have occasion to use the outcomes of relativity in some subsequent chapters of this text.

1.2 THE PRINCIPLE OF RELATIVITY

To describe a physical event, it is necessary to establish a frame of reference, such as one that is fixed in the laboratory. You should recall from your studies in mechanics that Newton's laws are valid in *all* inertial frames of reference. Because an inertial frame of reference is defined as one in which Newton's first law is valid, one can say that *an inertial frame of reference or an inertial system is a system in which a free body is not accelerating.* Furthermore, any system moving with constant velocity with respect to an inertial system must also be an inertial system. There is no preferred frame of reference. This means that the laws that describe an experiment performed in a vehicle moving with uniform velocity will be identical to the laws that describe the same experiment performed in another reference frame.

Inertial frame of reference

According to the **principle of Newtonian relativity,** the laws of mechanics must be the same in all inertial frames of reference. For example, if you perform an experiment while at rest in a laboratory, and an observer in a passing truck moving with constant velocity performs the same experiment, the observations of the person in the moving truck must agree with your observations. Specifically, in the laboratory or in the truck a ball thrown up rises and returns to the thrower's hand. Moreover, both events are measured to take the same time in the truck or in the laboratory, and Newton's second law may be used in both frames to compute this time. Although these experiments may look different to different observers (see Fig. 1.1, in which the Earth observer sees a different path for the ball), all observers agree on the validity of Newton's laws and principles such as conservation of energy and conservation of momentum. This implies that no experiment involving the laws of mechanics can detect any difference between the two inertial frames. The only thing that can be detected is the relative motion of one frame with respect to the other. That is, the notion of *absolute* motion through space is meaningless, as is the notion of a preferred reference frame.

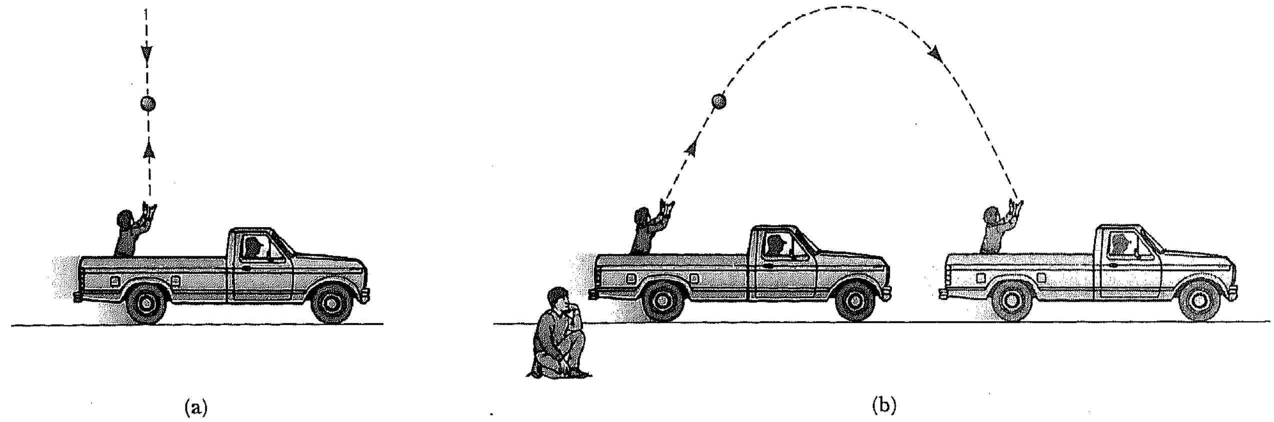


Figure 1.1 (a) The observer in the truck sees the ball move in a vertical path when thrown upward. (b) The Earth observer views the path of the ball as a parabola.

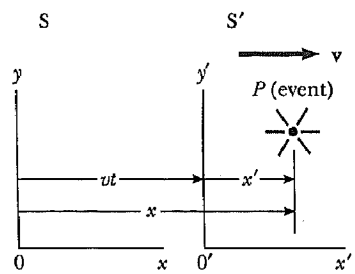


Figure 1.2 An event occurs at a point P. The event is observed by two observers in inertial frames S and S′, in which S′ moves with a velocity $\mathbf{v}$ relative to S.

Suppose that some physical phenomenon such as a light bulb flash, which we call an event, occurs in an inertial system. The event's location and time of occurrence can be specified by the coordinates (x, y, z, t). We would like to be able to transform the space and time coordinates of the event from one inertial system to another moving with uniform relative velocity. This is accomplished by using the so-called *Galilean transformation,* which owes its origin to Galileo.

Consider two inertial systems S and S′, as in Figure 1.2. The system S′ moves with a constant velocity $\mathbf{v}$ along the xx' axes, where $\mathbf{v}$ is measured relative to the system S. Clocks in S and S′ are synchronized; hence the origins of S and S′ coincide at $t = t' = 0$. We assume that an event occurs at the point P. An observer in the system S would describe the event with space-time coordinates (x, y, z, t), whereas an observer in system S′ would use (x', y', z', t') to describe the same event. As we can see from Figure 1.2, these coordinates are related by the equations

Galilean transformation of coordinates

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{1.1}$$

These equations constitute what is known as a **Galilean transformation of coordinates.** Note that the fourth coordinate, time, is *assumed* to be the same in both inertial systems. That is, within the framework of classical mechanics, all clocks run at the same rate regardless of their velocity, so that the time at which an event occurs for an observer in S is the same as the time for the same event in S′. Consequently, the time interval between two successive events should be the same for both observers. Although this assumption may seem obvious, it turns out to be *incorrect* when treating situations in which v is comparable to the speed of light. In fact, this point represents one of the most profound differences between Newtonian concepts and the ideas contained in Einstein's theory of relativity.

Now suppose two events are separated by a distance dx and a time interval dt as measured by an observer in S. It follows from Equations 1.1 that the corresponding displacement dx' measured by an observer in S′ is given by $dx' = dx - v\,dt$, where dx is the displacement measured by an observer in S. Because $dt = dt'$, we find that

$x' = x - vt$

$$\frac{dx'}{dt} = \frac{dx}{dt} - v$$

$\frac{dx'}{dt} = \frac{dx}{dt} - V$

or

$$u_x' = u_x - v \tag{1.2}$$

Galilean addition law for velocities

where u_x and u_x' are the instantaneous velocities of the object relative to S and S′, respectively.

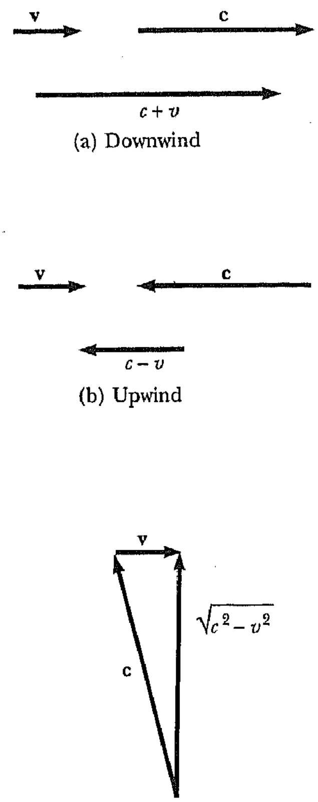
This result, which is called the **Galilean addition law for velocities** (or Galilean velocity transformation), is used in everyday observations and is consistent with our intuitive notions of time and space. However, we will soon see that these results from mechanics lead to serious contradictions when applied to electromagnetic waves.

The Speed of Light

It is natural to ask whether the concept of Newtonian relativity and the Galilean addition law for velocities in mechanics also apply to electricity, magnetism, and optics. Recall that Maxwell in the 1860s showed that the speed of light in free space was given by $c = (\mu_0\epsilon_0)^{-1/2} = 3.00 \times 10^8$ m/s. Physicists of the late 1800s thought that light waves (like sound and water waves) required a definite medium in which to move called the *ether*[2] and that the speed of light was c only in a special, absolute frame at rest with respect to the ether. In any other frame moving at speed v relative to the absolute ether frame, the Galilean addition law was expected to hold. Thus, the speed of light in this other frame was expected to be $c - v$ for light traveling in the same direction as the frame, $c + v$ for light traveling opposite to the frame, and in between these two values for light moving in an arbitrary direction with respect to the moving frame.

Because the existence of a preferred, absolute ether frame would show that light was similar to other classical waves (in requiring a medium) and that Newtonian ideas of an absolute or special coordinate frame were true, considerable importance was attached to establishing the existence of the ether frame. Because the speed of light is enormous, experiments involving light traveling in media moving at then attainable laboratory speeds had not been capable of detecting small changes of the size of $c \pm v$ prior to the late 1800s. Scientists of the period, realizing that the Earth moved rapidly around the Sun at 30 km/s, shrewdly decided to use the Earth itself as the moving frame in an

[2] It was proposed by Maxwell that light and other electromagnetic waves were waves in a luminiferous ether, which was present everywhere, even in empty space, but which allowed planets and other massive objects to pass freely through it without resistance.



(a) Downwind

(b) Upwind

(c) Across wind

Figure 1.3 If the velocity of the ether wind relative to the Earth is **v**, and **c** is the velocity of light relative to the ether, the speed of light relative to the Earth is (a) $c + v$ in the downwind direction, (b) $c - v$ in the upwind direction, and (c) $(c^2 - v^2)^{1/2}$ in the direction perpendicular to the wind.

attempt to improve their chances of detecting these small changes in light velocity.

From our point of view of observers fixed on Earth, we may say that we are stationary and that the absolute ether frame containing the medium for light propagation moves past us with speed v. Determining the speed of light under these circumstances is just like determining the speed of an aircraft in a moving air current or wind, and consequently we speak of an "ether wind" blowing through our apparatus fixed to the Earth. A direct method for detecting an ether wind would be to measure its influence on the speed of light determined by an apparatus fixed in a frame of reference on Earth. If $\mathbf{v}$ is the velocity of the ether relative to the Earth, then the speed of light should have its maximum value, $c + v$, when propagating downwind as shown in Figure 1.3a. Likewise, the speed of light should have its minimum value, $c - v$, when propagating upwind as in Figure 1.3b, and some intermediate value, $(c^2 - v^2)^{1/2}$, in the direction perpendicular to the ether wind as in Figure 1.3c. If the Sun is assumed to be at rest in the ether, then the velocity of the ether wind would be equal to the orbital velocity of the Earth around the Sun, which has a magnitude of about 3×10^4 m/s compared to $c = 3 \times 10^8$ m/s. Thus, the change in the speed of light would be about 1 part in 10^4 for measurements in the upwind or downwind directions, and changes of this size should be detectable. However, as we show in the next section, all attempts to detect such changes and establish the existence of the ether (and hence of the absolute frame) proved futile!

1.3 THE MICHELSON-MORLEY EXPERIMENT

The most famous experiment designed to detect small changes in the speed of light was performed in 1887 by Albert A. Michelson (1852–1931) and Edward W. Morley (1838–1923).[3] We should state at the outset that the outcome of the experiment was *negative,* thus contradicting the ether hypothesis. The experiment was designed to determine the velocity of the Earth with respect to the hypothetical ether. The experimental tool these pioneers used was the Michelson interferometer shown in Figure 1.4. One of the arms of the interferometer was aligned along the direction of the motion of the Earth through the ether. The Earth moving through the ether would be equivalent to the ether flowing past the Earth in the opposite direction with speed v as shown in Figure 1.4. This ether wind blowing in the opposite direction should cause the speed of light measured in the Earth's frame of reference to be $c - v$ as it approaches the mirror M_2 in Figure 1.4 and $c + v$ after reflection. The speed v is the speed of the Earth through space, and hence the speed of the ether wind, and c is the speed of light in the ether frame. The two beams of light reflected from M_1 and M_2 would recombine, and an interference pattern consisting of alternating dark and bright bands, or fringes, would be formed.

During the experiment, the interference pattern was observed while the interferometer was rotated through an angle of 90°. This rotation would change the speed of the ether wind along the direction of the arms of the interferometer. The effect of this rotation should have been to cause the fringe pattern to shift slightly but measurably. Measurements failed to show any change in the

[3] A. A. Michelson and E. W. Morley, *Am. J. Sci.* 134:333, 1887.

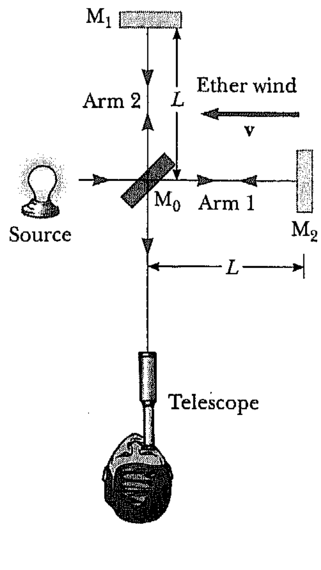


Figure 1.4 Diagram of the Michelson interferometer. According to the ether wind concept, the speed of light should be $c - v$ as the beam approaches mirror M_2 and $c + v$ after reflection.

interference pattern! The Michelson-Morley experiment was repeated by other researchers under various conditions and at different times of the year when the ether wind was expected to have changed direction and magnitude, but the results were always the same: *no fringe shift of the magnitude required was ever observed.*[4]

The negative results of the Michelson-Morley experiment not only contradicted the ether hypothesis, they also meant that it was impossible to measure the absolute velocity of the Earth with respect to the ether frame. However, as we shall see in the next section, Einstein offered a postulate for his theory of relativity that places quite a different interpretation on the null results. In later years when more was known about the nature of light, the idea of an ether that permeates all of space was relegated to the ash heap of worn-out concepts. Light is now understood to be a phenomenon that *requires no medium for its propagation.* As a result, the idea of an ether in which these waves could travel became unnecessary.

Details of the Michelson-Morley Experiment

To understand the outcome of the Michelson-Morley experiment, let us assume that the interferometer shown in Figure 1.4 has two arms of equal length L. First consider the beam traveling parallel to the direction of the ether wind, which is taken to be horizontal in Figure 1.4. According to Newtonian mechanics, as the beam moves to the right, its speed is reduced by the wind and its speed with respect to the Earth is $c - v$. On its return journey, as the light beam moves to the left downwind, its speed with respect to the Earth is $c + v$. Thus, the time of travel to the right is $L/(c - v)$, and the time of travel to the

[4] From an Earth observer's point of view, changes in the Earth's speed and direction in the course of a year are viewed as ether wind shifts. In fact, even if the speed of the Earth with respect to the ether were zero at some point in the Earth's orbit, six months later the speed of the Earth would be 60 km/s with respect to the ether, and one should find a clear fringe shift. None has ever been observed, however.

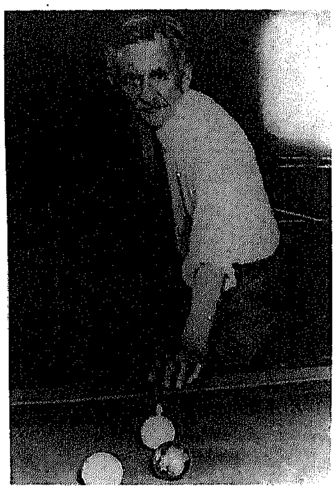


Figure 1.5 Albert A. Michelson (1852–1931). A German-American physicist, Michelson invented the interferometer and spent much of his life making accurate measurements of the speed of light. In 1907 he was the first American to be awarded the Nobel prize, which he received for his work in optics. His most famous experiment, conducted with Edward Morley in 1887, implied that it was impossible to measure the absolute velocity of the Earth with respect to the ether. Subsequent work by Einstein in his special theory of relativity eliminated the ether concept by assuming that the speed of light has the same value in all inertial reference frames. *(SPL/Photo Researchers)*

left is $L/(c + v)$. The total time of travel for the round trip along the horizontal path is

$$t_1 = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2Lc}{c^2 - v^2} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

Now consider the light beam traveling perpendicular to the wind, which is the vertical direction in Figure 1.4. Because the speed of the beam relative to the Earth is $(c^2 - v^2)^{1/2}$ in this case (see Fig. 1.3c) then, the time of travel for each half of this trip is $L/(c^2 - v^2)^{1/2}$, and the total time of travel for the round trip is

$$t_2 = \frac{2L}{(c^2 - v^2)^{1/2}} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Thus, the time difference between the light beam traveling horizontally and the beam traveling vertically is

$$\Delta t = t_1 - t_2 = \frac{2L}{c}\left[\left(1 - \frac{v^2}{c^2}\right)^{-1} - \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right]$$

Because $v^2/c^2 \ll 1$, this expression can be simplified by using the following binomial expansions after dropping all terms higher than second order:

$$(1 - x)^n \approx 1 - nx \qquad (\text{for } x \ll 1)$$

In our case, $x = v^2/c^2$, and we find

$$\Delta t = t_1 - t_2 \approx \frac{Lv^2}{c^3} \tag{1.3}$$

The two light beams start out in phase and return to form an interference pattern. Let us assume that the interferometer is adjusted for parallel fringes and that a telescope is focused on one of these fringes. The time difference between the two light beams gives rise to a phase difference between the beams, producing an interference pattern when they combine at the position of the telescope. A difference in the pattern should be detected by rotating the interferometer through 90° in a horizontal plane, such that the two beams exchange roles (Fig. 1.6). This results in a net time difference of twice that given by

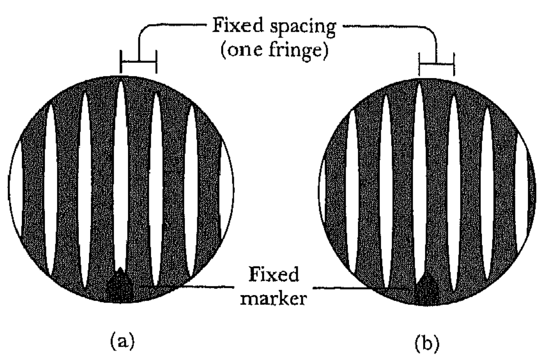


Figure 1.6 Interference fringe schematic showing (a) fringes before rotation and (b) expected fringe shift after a rotation of the interferometer by 90°.

Equation 1.3. Thus, the path difference corresponding to this time difference is

$$\Delta d = c(2\,\Delta t) = \frac{2Lv^2}{c^2}$$

The corresponding fringe shift is equal to this path difference divided by the wavelength of light, λ, because a change in path of 1 wavelength corresponds to a shift of one fringe.

$$\text{Shift} = \frac{2Lv^2}{\lambda c^2} \tag{1.4}$$

In the experiments by Michelson and Morley, each light beam was reflected by mirrors many times to give an increased effective path length L of about 11 m. Using this value, and taking v to be equal to 3×10^4 m/s, the speed of the Earth about the Sun, gives a path difference of

$$\Delta d = \frac{2(11\text{ m})(3 \times 10^4\text{ m/s})^2}{(3 \times 10^8\text{ m/s})^2} = 2.2 \times 10^{-7}\text{ m}$$

This extra distance of travel should produce a noticeable shift in the fringe pattern. Specifically, using light of wavelength 500 nm, we find a fringe shift for rotation through 90° of

$$\text{Shift} = \frac{\Delta d}{\lambda} = \frac{2.2 \times 10^{-7}\text{ m}}{5.0 \times 10^{-7}\text{ m}} \approx 0.40$$

The instrument used by Michelson and Morley had the capability of detecting a shift in the fringe pattern as small as 0.01 fringe. However, *they detected no shift in the fringe pattern.* Since then, the experiment has been repeated many times by various scientists under various conditions, and no fringe shift has ever been detected. Thus, it was concluded that one cannot detect the motion of the Earth with respect to the ether. fringe

Many efforts were made to explain the null results of the Michelson-Morley experiment and to save the absolute ether frame concept and the Galilean addition law for the velocity of light. Because all these proposals have been shown to be wrong, we consider them no further here and turn instead to an auspicious proposal made by George F. Fitzgerald and Hendrik A. Lorentz. In the 1890s, Fitzgerald and Lorentz tried to explain the null results by making the following ad hoc assumption. They proposed that the length of an object moving at speed v would contract along the direction of travel by a factor of $\sqrt{1 - v^2/c^2}$. The net result of this contraction would be a change in length of one of the arms of the interferometer such that no path difference would occur as it was rotated.

Never in the history of physics were such valiant efforts devoted to trying to explain the absence of an expected result as those directed at the Michelson-Morley experiment. The stage was set for the brilliant Albert Einstein, who solved the problem in 1905 with his special theory of relativity.

1.4 POSTULATES OF SPECIAL RELATIVITY

In the previous section we noted the impossibility of measuring the speed of the ether with respect to the Earth and the failure of the Galilean addition law for speeds near that of light. In 1905, Albert Einstein (Fig. 1.7) proposed a theory that boldly removed these difficulties and at the same time completely altered our notion of space and time.[5] Einstein based his special theory of relativity on two postulates.

Postulates of special relativity

1. **The Principle of Relativity:** All the laws of physics are the same in all inertial reference frames.
2. **The Constancy of the Speed of Light:** The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

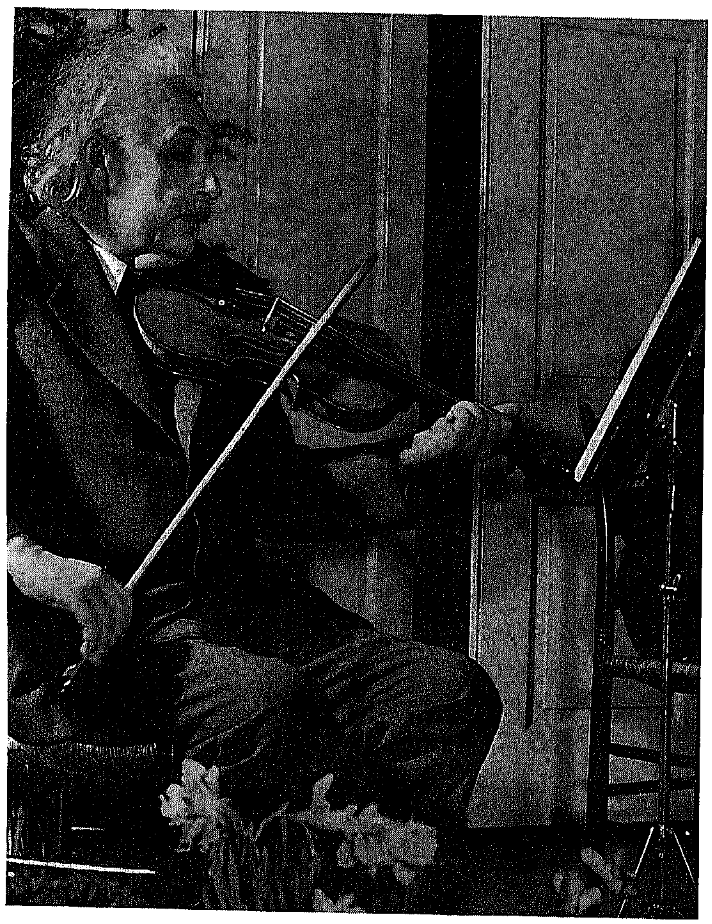
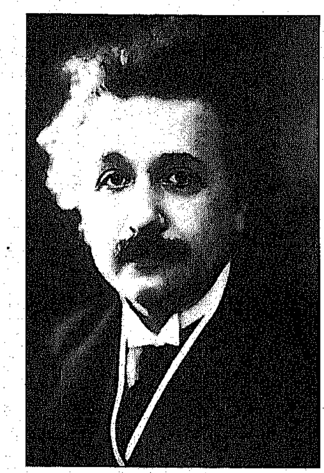


Figure 1.7 Albert Einstein playing his beloved fiddle, 1941. *(Hansel Micth, Life Magazine © Time, Inc.)*

[5] A. Einstein, "On the Electrodynamics of Moving Bodies," *Ann. Physik* 17: 891, 1905. For an English translation of this article and other publications by Einstein, see the book by H. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity*, Dover, 1958.

The first postulate asserts that *all* the laws of physics, those dealing with electricity and magnetism, optics, thermodynamics, mechanics, and so on, will be the same in all coordinate frames moving with constant velocity relative to one another. This postulate is a sweeping generalization of Newton's concept of relativity that refers only to the laws of mechanics. From an experimental point of view, Einstein's principle of relativity means that any kind of experiment (measuring the velocity of light for example) performed in one laboratory at rest must give the same result when performed in another laboratory moving at constant velocity relative to the first one. Hence, no preferred inertial system exists, and it is impossible to detect absolute motion.

Note that postulate 2, the principle of the constancy of the speed of light, is consistent with postulate 1: If the speed of light was not the same in all inertial frames but was c in only one, it would be possible to distinguish between inertial frames, and one could identify a preferred, absolute frame in contradiction to postulate 1. Postulate 2 also does away with the problem of measuring the speed of the ether by essentially denying the existence of the ether and boldly asserting that light always moves with speed c with respect to any inertial observer whatsoever.



B I O G R A P H Y

ALBERT EINSTEIN
(1879–1955)

Albert Einstein, one of the greatest physicists of all time, was born in Ulm, Germany. As a child, Einstein was very unhappy with the discipline of German schools and completed his early education in Switzerland at age 16. Because he was unable to obtain an academic position following graduation from the Swiss Federal Polytechnic School in 1901, he accepted a job at the Swiss Patent Office in Berne. During his spare time, he continued his studies in theoretical physics. In 1905, at the age of 26, he published four scientific papers that revolutionized physics. One of these papers, which won him the Nobel prize in 1921, dealt with the photoelectric effect. Another was concerned with Brownian motion, the irregular motion of small particles suspended in a liquid. The remaining two papers were concerned with what is now considered his most important contribution of all, the special theory of relativity. In 1915, Einstein published his work on the general theory of relativity, which relates gravity to the structure of space and time. One of the remarkable predictions of the theory is that strong gravitational forces in the vicinity of very massive objects cause light beams to deviate from a straight-line path. This and other predictions of the general theory of relativity have been experimentally verified (see the essay in this chapter by Clifford Will).

Einstein made many other important contributions to the development of modern physics, including the concept of the light quantum and the idea of stimulated emission of radiation, which led to the invention of the laser 40 years later. However, throughout his life he rejected the probabilistic interpretation of quantum mechanics when describing events on the atomic scale, in favor of a deterministic view. He is quoted as saying "God does not play dice with nature." This comment is reputed to have been answered by Niels Bohr, one of the founders of quantum mechanics, with "Don't tell God what to do!"

In 1933, Einstein left Germany under the power of the Nazis and spent his remaining years at the Institute for Advanced Study in Princeton, New Jersey. He devoted most of his later years to an unsuccessful search for a unified theory of gravity and electromagnetism.

Photo courtesy of AIP/Niels Bohr Library

Although the Michelson-Morley experiment was performed before Einstein published his work on relativity, it is not clear that Einstein was aware of the details of the experiment. Nonetheless, the null result of the experiment can be readily understood within the framework of Einstein's theory. According to his principle of relativity, the premises of the Michelson-Morley experiment were incorrect. In the process of trying to explain the expected results, we stated that when light traveled against the ether wind its speed was $c - v$, in accordance with the Galilean addition law for velocities. However, if the state of motion of the observer or of the source has no influence on the value found for the speed of light, one will always measure the value to be c. Likewise, the light makes the return trip after reflection from the mirror at a speed of c, and not with the speed $c + v$. Thus, the motion of the Earth should not influence the fringe pattern observed in the Michelson-Morley experiment, and a null result should be expected.[6]

If we accept Einstein's theory of relativity, we must conclude that relative motion is unimportant when measuring the speed of light. At the same time, we must alter our common sense notions of space and time and be prepared for some rather bizarre consequences.

1.5 CONSEQUENCES OF SPECIAL RELATIVITY

Almost everyone who has dabbled even superficially with science is aware of some of the startling predictions that arise because of Einstein's approach to relative motion. As we examine some of the consequences of relativity in this section, we shall find that they conflict with our basic notions of space and time. We restrict our discussion to the concepts of length, time, and simultaneity, which are quite different in relativistic mechanics and Newtonian mechanics. For example, we will find that *the distance between two points and the time interval between two events depend on the frame of reference in which they are measured.* That is, *there is no such thing as absolute length or absolute time in relativity*. Furthermore, *events at different locations that occur simultaneously in one frame are not simultaneous in another frame moving uniformly past the first.*

Before we discuss the consequences of special relativity, we must first understand how an observer in an inertial reference frame describes an event. We define an event as an occurrence described by three space coordinates and one time coordinate. In general, different observers in different inertial frames would describe the same event with different space-time coordinates.

The reference frame used to describe an event consists of a coordinate grid and a set of clocks situated at the grid intersections, as shown in Figure 1.8 in two dimensions. It is necessary that the clocks be synchronized. This can be

[6] Perhaps at this point you have rightly concluded that the Galilean transformation is incorrect; that is, it does not keep all the laws of physics in the same form for a frame at rest and one in uniform motion. The correct coordinate and time transformations that preserve the form of all physical laws in two coordinate systems moving uniformly with respect to each other are called Lorentz transformations. These are derived in Section 1.6. Although the Galilean transformation preserves the form of Newton's laws in two frames moving uniformly with respect to each other, Newton's laws of mechanics are limited laws that are only valid for low speeds. In general, Newton's laws must be replaced by Einstein's relativistic laws of mechanics, which hold for all speeds and are invariant under the Lorentz transformations.

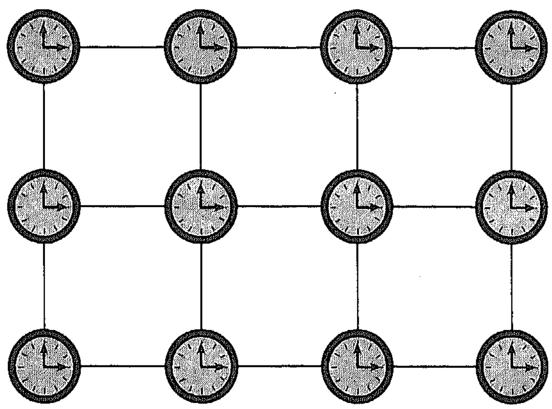


Figure 1.8 In relativity, we use a reference frame consisting of a coordinate grid and a set of synchronized clocks.

accomplished in many ways with the help of light signals. For example, suppose an observer at the origin with a master clock sends out a pulse of light at $t = 0$. The light pulse takes a time r/c to reach a second clock, situated a distance r from the origin. Hence, the second clock will be synchronized with the clock at the origin if the second clock reads a time r/c at the instant the pulse reaches it. This procedure of synchronization assumes that the speed of light has the same value in all directions and in all inertial frames. Furthermore, the procedure concerns an event recorded by an observer in a specific inertial reference frame. Clocks in other inertial frames can be synchronized in a similar manner. An observer in some other inertial frame would assign different spacetime coordinates to events, using another coordinate grid with another array of clocks.

Simultaneity and the Relativity of Time

A basic premise of Newtonian mechanics is that a universal time scale exists that is the same for all observers. In fact, Newton wrote that "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external." Thus, Newton and his followers simply took simultaneity for granted. In his special theory of relativity, Einstein abandoned this assumption. According to Einstein, *a time interval measurement depends on the reference frame in which the measurement is made.*

Einstein devised the following thought experiment to illustrate this point. A boxcar moves with uniform velocity, and two lightning bolts strike the ends of the boxcar, as in Figure 1.9a, leaving marks on the boxcar and ground. The marks left on the boxcar are labeled A' and B'; those on the ground are labeled A and B. An observer at O' moving with the boxcar is midway between A' and B', and a ground observer at O is midway between A and B. The events recorded by the observers are the light signals from the lightning bolts.

The two light signals reach the observer at O at the same time, as indicated in Figure 1.9b. This observer realizes that the light signals have traveled at the same speed over equal distances. Thus, observer O concludes that the events at A and B occurred simultaneously. Now consider the same events as viewed by the observer on the boxcar at O'. By the time the light has reached observer O, observer O' has moved as indicated in Figure 1.9b. Thus, the light signal

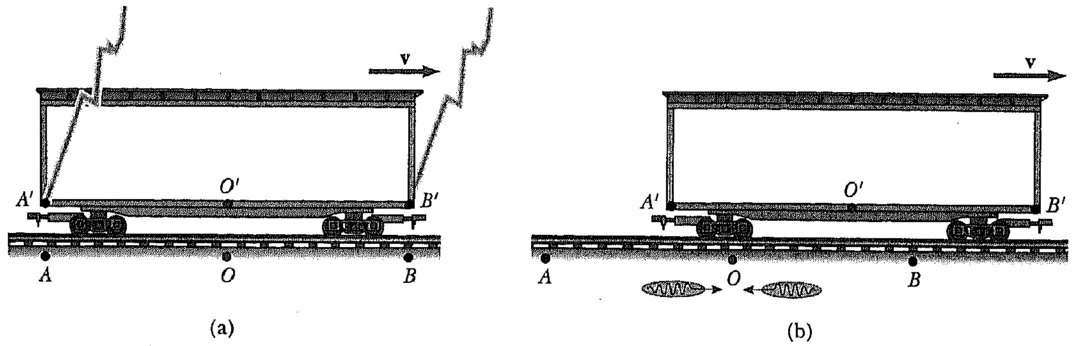


Figure 1.9 Two lightning bolts strike the ends of a moving boxcar. (a) The events appear to be simultaneous to the stationary observer at O, who is midway between A and B. (b) The events do not appear to be simultaneous to the observer at O', who claims that the front of the train is struck *before* the rear.

from B' has already swept past O', but the light from A' has not yet reached O'. According to Einstein, *observer O' must find that light travels at the same speed as that measured by observer O.* Therefore, observer O' concludes that the lightning struck the front of the boxcar before it struck the back. This thought experiment clearly demonstrates that the two events, which appear to O to be simultaneous, do not appear to O' to be simultaneous. In other words,

> two events that are simultaneous in one frame are in general not simultaneous in a second frame moving with respect to the first. That is, simultaneity is not an absolute concept, but one that depends on the state of motion of the observer.

At this point, you might wonder which observer is right concerning the two events. The answer is that *both are correct,* because the principle of relativity states that *there is no preferred inertial frame of reference.* Although the two observers reach different conclusions, both are correct in their own reference frame because the concept of simultaneity is not absolute. This, in fact, is the central point of relativity—any uniformly moving frame of reference can be used to describe events and do physics. However, observers in different inertial frames of reference will always measure different time intervals with their clocks and different distances with their meter sticks. Nevertheless, they will both agree on the forms of the laws of physics in their respective frames, because these laws must be the same for all observers in uniform motion. It is the alteration of time and space that allows the laws of physics (including Maxwell's equations) to be the same for all observers in uniform motion.

Time Dilation

The fact that observers in different inertial frames always measure different time intervals between a pair of events can be illustrated in another way by a vehicle moving to the right with a speed v, as in Figure 1.10a. A mirror is fixed to the ceiling of the vehicle, and observer O', at rest in this system, holds a laser a distance d below the mirror. At some instant the laser emits a pulse of light directed toward the mirror (event 1), and at some later time, after reflecting from the mirror, the pulse arrives back at the laser (event 2). Observer O'

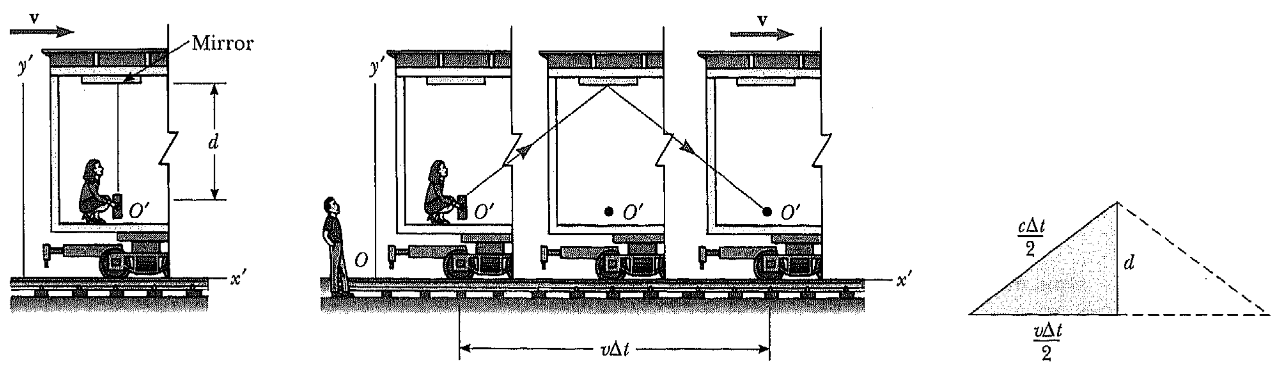


Figure 1.10 (a) A mirror is fixed to a moving vehicle, and a light pulse leaves O' at rest in the vehicle. (b) Relative to a stationary observer on Earth, the mirror and O' move with a speed v. Note that the distance the pulse travels measured by the stationary observer on Earth is greater than $2d$. (c) The right triangle for calculating the relationship between Δt and $\Delta t'$.

carries a clock, C', which she uses to measure the time interval $\Delta t'$ between these two events. Because the light pulse has the speed c, the time it takes to travel from O' to the mirror and back can be found from the definition of speed:

$$\Delta t' = \frac{\text{distance traveled}}{\text{speed of light}} = \frac{2d}{c} \tag{1.5}$$

This time interval $\Delta t'$—measured by O', who, remember, is at rest in the moving vehicle—requires only a *single* clock, C', in this reference frame.

Now consider the same set of events as viewed by an observer O in a different frame (Fig. 1.10b). According to this observer, the mirror and laser are moving to the right with a speed v. The sequence of events would appear entirely different as viewed by this observer. By the time the light from the laser reaches the mirror, the mirror will have moved a distance $v\,\Delta t/2$, where Δt is the time for the light pulse to travel from O' to the mirror and back as measured by the second observer O. In other words, the second observer concludes that, because of the motion of the clock, if the light is to hit the mirror, it must leave the laser at an angle with respect to the vertical direction. Comparing Figures 1.10a and 1.10b, we see that the light must travel farther in the second frame than in the first frame.

According to the second postulate of relativity, both observers must measure c for the speed of light. Because the light travels farther in the second frame, it follows that the time interval Δt measured by an observer in the second frame is longer than the time interval $\Delta t'$ measured by an observer in the first frame. To obtain a relationship between Δt and $\Delta t'$, it is convenient to use the right triangle shown in Figure 1.10c. The Pythagorean theorem gives

$$\left(\frac{c\,\Delta t}{2}\right)^2 = \left(\frac{v\,\Delta t}{2}\right)^2 + d^2$$

Solving for Δt gives

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{1 - v^2/c^2}} \tag{1.6}$$

Because $\Delta t' = 2d/c$, we can express Equation 1.6 as

Time dilation

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - (v^2/c^2)}} = \gamma\,\Delta t' \tag{1.7}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. This result says that the time interval Δt measured by the observer moving with respect to the clock is *longer* than the time interval $\Delta t'$ measured by the observer at rest with respect to the clock *because γ is always greater than unity.* This effect is known as time dilation.

A moving clock runs slower

The time interval $\Delta t'$ in Equation 1.7 is called the proper time. In general, **proper time** is defined as *the time interval between two events as measured by an observer who sees the events occur at the same point in space.* In our case, observer O' measures the proper time. That is, *proper time is always the time measured by an observer moving along with the clock.*

Because the time between ticks of a moving clock, $\gamma(2d/c)$, is observed to be longer than the time between ticks of an identical clock at rest, $2d/c$, one often says, *"A moving clock runs slower than a clock at rest by a factor of γ."* This is true for ordinary mechanical clocks as well as for the light clock just described. In fact, we can generalize these results by stating that *all physical processes, including chemical reactions and biological processes, slow down relative to a stationary clock when they occur in a moving frame.* For example, the heartbeat of an astronaut moving through space would keep time with a clock inside the spaceship, but both the astronaut's clock and her heartbeat appear slow to an observer, with another clock, in any other reference frame. The astronaut would not have any sensation of life slowing down in her frame.

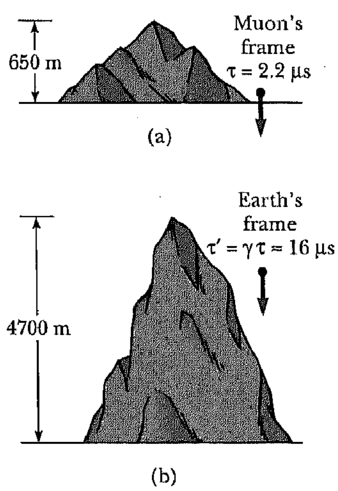


Figure 1.11 (a) Muons traveling with a speed of $0.99c$ travel only about 650 m as measured in the muons' reference frame, where their lifetime is about 2.2 μs. (b) The muons travel about 4700 m as measured by an observer on Earth. Because of time dilation, the muons' lifetime is longer as measured by the Earth observer.

Time dilation is a very real phenomenon that has been verified by various experiments. For example, muons are unstable elementary particles, which have a charge equal to that of an electron and a mass 207 times that of the electron. Muons can be produced by the collision of cosmic radiation with atoms high in the atmosphere. Muons have a lifetime of only 2.2 μs when measured in a reference frame at rest with respect to them. If we take 2.2 μs as the average lifetime of a muon and assume that its speed is close to the speed of light, we would find that these particles could travel a distance of about 650 m before they decayed. Hence, they could not reach the Earth from the upper atmosphere where they are produced. However, experiments show that a large number of muons *do* reach the Earth. The phenomenon of time dilation explains this effect (see Fig. 1.11a). Relative to an observer on Earth, the muons have a lifetime equal to $\gamma\tau$, where $\tau = 2.2$ μs is the lifetime in a frame of reference traveling with the muons. For example, for $v = 0.99c$, $\gamma \approx 7.1$ and $\gamma\tau \approx 16$ μs. Hence, the average distance traveled as measured by an observer on Earth is $\gamma v \tau \approx 4700$ m, as indicated in Figure 1.11b.

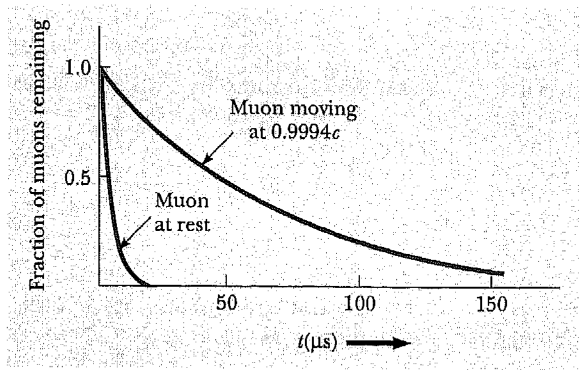


Figure 1.12 Decay curves for muons traveling at a speed of $0.9994c$ and for muons at rest.

In 1976, experiments with muons were conducted at the laboratory of the European Council for Nuclear Research (CERN) in Geneva. Muons were injected into a large storage ring, reaching speeds of about $0.9994c$. Electrons produced by the decaying muons were detected by counters around the ring, enabling scientists to measure the decay rate, and hence the lifetime of the muons. The lifetime of the moving muons was measured to be about 30 times as long as that of the stationary muon (see Fig. 1.12), in agreement with the prediction of relativity to within two parts in a thousand.

The results of an experiment reported by Hafele and Keating provided direct evidence for the phenomenon of time dilation.[7] The experiment involved the use of very stable cesium beam atomic clocks. Time intervals measured with four such clocks in jet flight were compared with time intervals measured by reference atomic clocks located at the U.S. Naval Observatory. To compare these results with the theory, many factors had to be considered, including periods of acceleration and deceleration relative to the Earth, variations in direction of travel, and the weaker gravitational field experienced by the flying clocks compared with the Earth-based clock. Their results were in good agreement with the predictions of the special theory of relativity and can be explained in terms of the relative motion between the Earth's rotation and the jet aircraft. In their paper, Hafele and Keating report the following: "Relative to the atomic time scale of the U.S. Naval Observatory, the flying clocks lost 59 ± 10 ns during the eastward trip and gained 273 ± 7 ns during the westward trip. . . . These results provide an unambiguous empirical resolution of the famous clock paradox with macroscopic clocks."

[7] J. C. Hafele and R. E. Keating, "Around the World Atomic Clocks: Relativistic Time Gains Observed," *Science*, July 14, 1972, p. 168.

EXAMPLE 1.1 What Is the Period of the Pendulum?

The period of a pendulum is measured to be 3.0 s in the rest frame of the pendulum. What is the period of the pendulum when measured by an observer moving at a speed of $0.95c$ with respect to the pendulum?

Solution In this case, the proper time is equal to 3.0 s. Instead of the observer moving at $0.95c$, we can take the equivalent point of view that the observer is at rest and that the pendulum is moving at $0.95c$ past the stationary observer. Hence the pendulum is an example of a moving clock. Because a moving clock runs slower than a stationary clock by γ, Equation 1.7 gives

$$T = \gamma T' = \frac{1}{\sqrt{1 - (0.95c)^2/c^2}} (3.0 \text{ s})$$

$$T' = (3.2)(3.0 \text{ s}) = 9.6 \text{ s}$$

That is, a moving pendulum slows down or takes longer to complete a period.

Length Contraction

We have seen that measured time intervals are not absolute, that is, the time interval between two events depends on the frame of reference in which it is measured. Likewise, the measured distance between two points depends on the frame of reference. The **proper length** of an object is defined as *the length of the object measured by someone who is at rest with respect to the object.* You should note that proper length is defined similarly to proper time, in that proper time is the time between ticks of a clock measured by an observer who is at rest with respect to the clock. The length of an object measured in a reference frame in which the object is moving is always less than the proper length. This effect is known as **length contraction.**

To understand length contraction quantitatively, let us consider a spaceship traveling with a speed v from one star to another, and two observers: one on Earth and the other in the spaceship. The observer at rest on Earth (and also assumed to be at rest with respect to the two stars) measures the distance between the stars to be L', where L' is the proper length. According to this observer, the time it takes the spaceship to complete the voyage is $\Delta t = L'/v$. What does an observer in the moving spaceship measure for the distance between the stars? Because of the time dilation, the space traveler measures a smaller time of travel: $\Delta t' = \Delta t/\gamma$. The space traveler claims to be at rest and sees the destination star as moving toward the spaceship with speed v. Because the space traveler reaches the star in the time $\Delta t'$, he or she concludes that the distance, L, between the stars is shorter than L. This distance measured by the space traveler is given by

$$L = v\,\Delta t' = v\frac{\Delta t}{\gamma}$$

Because $L' = v\,\Delta t$, we see that $L = L'/\gamma$ or

Length contraction

$$L = L'\left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{1.8}$$

where $(1 - v^2/c^2)^{1/2}$ is a factor less than one. This result may be interpreted as follows:

If an object has a length L' when it is at rest, then when measured by an observer moving with relative speed of v in a direction parallel to its length, it will appear to contract to the length L, where $L = L'(1 - v^2/c^2)^{1/2}$.

Note that the length contraction takes place only along the direction of motion. For example, suppose a stick moves past a stationary Earth observer with a speed v as in Figure 1.13. The length of the stick as measured by an observer in the frame attached to it is the proper length L', as illustrated in Figure 1.13a. The length of the stick, L, as measured by the Earth observer in the stationary frame is shorter than L' by the factor $(1 - v^2/c^2)^{1/2}$. Note that length contraction is a symmetric effect: If the stick were at rest on Earth, an observer in the moving frame would also measure its length to be shorter by the same factor $(1 - v^2/c^2)^{1/2}$.

It is important to emphasize a particular type of experiment for which $L = vt$. Let us return to the example of the decaying muons moving at speeds close to the speed of light. An observer in the muon's reference frame would measure the proper lifetime, whereas an Earth-based observer measures the proper height of the mountain in Figure 1.11. In the muon's reference frame, there is no time dilation, but the distance of travel is observed to be shorter when measured in this frame. Likewise, in the Earth observer's reference frame, there is time dilation, but the distance of travel is measured to be the proper height of the mountain. Thus, when calculations on the muon are performed in both frames, one sees the effect of "offsetting penalties," and the outcome of the experiment is the same! Note that proper length and proper time are measured in *different* reference frames.

If an object in the shape of a box passing by could be photographed, its image would show length contraction, but its shape would also be distorted. This is illustrated in the computer-simulated drawings shown in Figure 1.14 for a box moving past an observer with a speed $v = 0.8c$. When the shutter of the camera is opened, it records the shape of the object at a given instant of time. Because light from different parts of the object must arrive at the shutter at the same time (when the photograph is taken), light from more distant parts of the object must start its journey earlier than light from closer parts. Hence, the photograph records different parts of the object at different times. This results in a highly distorted image, which shows horizontal length contraction, vertical curvature, and image rotation.

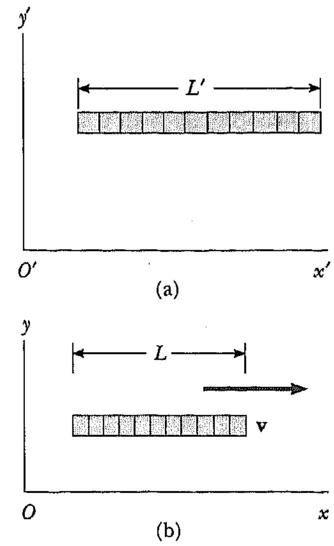


Figure 1.13 A stick moves to the right with a speed v. (a) The stick as viewed in a frame attached to it. (b) The stick as seen by an observer who sees it move past her at v. Any inertial observer finds that the length of a meter stick moving past her with speed v is less than the length of a stationary stick by a factor of $(1 - v^2/c^2)^{1/2}$.

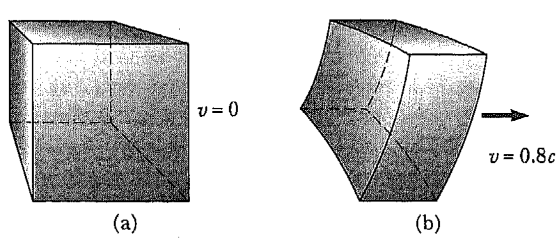


Figure 1.14 Computer-simulated photographs of a box (a) at rest relative to the camera and (b) moving at a speed $v = 0.8c$ relative to the camera.

EXAMPLE 1.2 The Contraction of a Spaceship

A spaceship is measured to be 100 m long while it is at rest with respect to an observer. If this spaceship now flies by the observer with a speed of $0.99c$, what length will the observer find for the spaceship?

Solution From Equation 1.8, the length measured as the spaceship flies by is

$$L = L'\sqrt{1-\frac{v^2}{c^2}} = (100 \text{ m})\sqrt{1-\frac{(0.99c)^2}{c^2}} = 14 \text{ m}$$

Exercise 1 If the ship moves past the observer at $0.01000c$, what length will the observer measure?

Answer 99.99 m.

EXAMPLE 1.3 How High Is the Spaceship?

An observer on Earth sees a spaceship at an altitude of 435 m moving downward toward the Earth at $0.970c$. What is the altitude of the spaceship as measured by an observer in the spaceship?

Solution The moving observer in the ship finds the altitude to be

$$L = L'\sqrt{1-\frac{v^2}{c^2}} = (435 \text{ m})\sqrt{1-\frac{(0.970c)^2}{c^2}} = 106 \text{ m}$$

EXAMPLE 1.4 The Triangular Spaceship

A spaceship in the form of a triangle flies by an observer at $0.950c$. When the ship is measured by an observer at rest with respect to the ship (Fig. 1.15a), the distances x and y are found to be 50.0 m and 25.0 m, respectively.

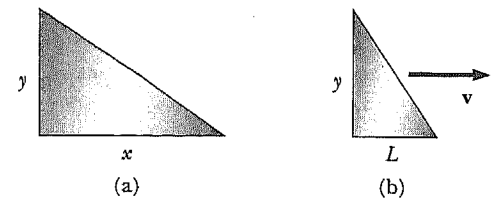


Figure 1.15 (Example 1.4) (a) When the spaceship is at rest, its shape is as shown. (b) The spaceship appears to look like this when it moves to the right with a speed v. Note that only its x dimension is contracted in this case.

What is the shape of the ship as seen by an observer who sees the ship in motion along the direction shown in Figure 1.15b?

Solution The observer sees the horizontal length of the ship to be contracted to a length of

$$L = L'\sqrt{1-\frac{v^2}{c^2}}$$
$$= (50.0 \text{ m})\sqrt{1-\frac{(0.950c)^2}{c^2}} = 15.6 \text{ m}$$

The 25-m vertical height is unchanged because it is perpendicular to the direction of relative motion between the observer and the spaceship. Figure 1.15b represents the shape of the spaceship as seen by the observer who sees the ship in motion.

OPTIONAL

THE TWIN PARADOX

An intriguing consequence of time dilation is the so-called clock or twin paradox. Consider an experiment involving a set of identical 20-year-old twins named Speedo and Goslo. The twins carry with them identical clocks that have been synchronized. Speedo, the more adventuresome of the two, sets out on an epic journey to planet X, which is 10 lightyears from Earth. Furthermore, his spaceship is capable of reaching a speed of $0.500c$ relative to the inertial frame of his twin brother. After reaching planet X, Speedo becomes homesick and impetuously sets out on a return trip to Earth at the same high speed he had attained on the outbound journey. On his return, Speedo is shocked to discover that many things have changed during his absence. To Speedo, the most significant change is that his twin brother Goslo *has aged more than he* and is now 60 years of age. Speedo, on the other hand, has aged by only 34.6 years. We see that the bodily processes continued for a longer period of time for one twin than for the other. The accelerated twin moved from one reference frame to another, so his accumulated time was less.

At this point, it is fair to raise the following question—Which twin is the traveler and which twin would really be the younger of the two? From Speedo's perspective, it is he who is at rest while Goslo is on a high-speed space journey. To Speedo, it is Goslo and the Earth that have raced away on a 17.3-year journey and then headed back for another 17.3 years. This leads to an apparent contradiction. Which twin will have developed the signs of excess aging?

To resolve this apparent paradox, recall that special relativity deals with inertial frames of reference moving with respect to one another at *uniform speed.* However, the trip situation is not symmetric. Speedo, the space traveler, must experience acceleration during his journey. As a result, his state of motion is not always uniform, and consequently Speedo is not in an inertial frame. He cannot regard himself to always be at rest and Goslo to be in uniform motion. Hence Speedo cannot simply apply time dilation to Goslo's motion, because to do so would be an incorrect application of special relativity. Therefore there is no paradox.

The conclusion that Speedo is in a noninertial frame is inescapable. We may diminish the length of time needed to accelerate and decelerate Speedo's spaceship to a negligible interval by using very large and expensive rockets and claim that he spends all but a negligible amount of time coasting to planet X at $0.500c$ in an inertial frame. However, to return to Earth, Speedo must slow down, reverse his motion, and return in a different inertial frame, one which is moving uniformly toward the Earth. At the very best, Speedo is in two different inertial frames. The important point here is that even when we idealize Speedo's trip, it consists of motion in two different inertial frames and a very real lurch as he hops from the outbound ship to the returning Earth shuttle. In other words, only Goslo remains in a single inertial frame, and so only he can correctly apply the simple time dilation formula to Speedo's trip. Thus, Goslo finds that instead of aging 40 years (20 lightyears/$0.500c$), Speedo actually ages only $(\sqrt{1 - v^2/c^2})$ (40 years), or 34.6 years. Clearly, Speedo spends 17.3 years going to planet X and 17.3 years returning in agreement with our earlier statement.

The result that Speedo ages 34.6 years while Goslo ages 40 years can be confirmed in a very direct experimental way by Speedo if we use the special theory of relativity but take into account the fact that *Speedo's trip takes place in two different inertial frames.* In yet another flight of fancy, suppose that Goslo celebrates his birthday each year in a flashy way, sending a powerful laser pulse to inform his twin by light flash that Goslo is another year older and wiser. Because Speedo is in an inertial frame on the outbound trip in which the Earth appears to be receding at $0.500c$, the flashes occur at a rate of one every

$$\frac{1}{\sqrt{1 - (v^2/c^2)}} \text{ year} = \frac{1}{\sqrt{1 - [(0.500c)^2/c^2]}} \text{ year} = 1.15 \text{ years}$$

This occurs because moving clocks run slower. Also, because the Earth is receding, each successive flash must travel an additional distance of $(0.500c)(1.15 \text{ years})$ between flashes. Consequently, Speedo observes flashes to arrive with a total time between flashes of $1.15 \text{ years} + (0.500c)(1.15 \text{ years})/c = 1.73$ years. The total number of flashes seen by Speedo on his outbound voyage is therefore $(1 \text{ flash}/1.73 \text{ years})(17.3 \text{ years}) = 10$ flashes. This means that Speedo views the Earth clocks to run more slowly than his own on the outbound trip because he observes 17.3 years to have passed for him while only 10 years have passed on Earth.

On the return voyage, because the Earth is racing toward Speedo with speed $0.500c$, successive flashes have less distance to travel, and the total time Speedo sees between the arrival of flashes is drastically shortened: $1.15 \text{ years} - (0.500)(1.15 \text{ years}) = 0.577$ year/flash. Thus, during the return trip, Speedo sees $(1 \text{ flash}/0.577 \text{ year})(17.3 \text{ years}) = 30$ flashes in total. In sum, during his 34.6 years of travel, Speedo receives (10 + 30) flashes, indicating that his twin has aged 40 years. Notice that there has been no failure of special relativity for Speedo as long as we take his *two* inertial frames into account. On both the outbound and inbound trips Speedo correctly judges the Earth clocks to run slower than his own, but on the return trip his rapid movement toward the light flashes more than compensates for the slower rate of flashing.

Figure 1.16 "I love hearing that lonesome wail of the train whistle as the frequency of the wave changes due to the Doppler effect."

The Relativistic Doppler Shift

Another important consequence of time dilation is the shift in frequency found for light emitted by atoms in motion as opposed to light emitted by atoms at rest. A similar phenomenon, the mournful drop in pitch of the sound of a passing train's whistle, known as the Doppler effect, is quite familiar to most cowboys (Fig. 1.16). The Doppler shift for sound is usually studied in introductory physics courses and is especially interesting because motion of the source with respect to the medium of propagation can be clearly distinguished from motion of the observer. This means that in the case of sound we can distinguish the "absolute motion" of frames moving with respect to the air, which is the medium of propagation for sound.

Light waves must be analyzed differently from sound, because light waves require no medium of propagation and no method exists of distinguishing the motion of the light source from the motion of the observer. Thus, we expect to find a different formula for the Doppler shift of light waves, one that is only sensitive to the *relative* motion of source and observer and that holds for relative speeds of source and observer approaching c.

Consider a source of light waves at rest in frame S, emitting waves of frequency f and wavelength λ as measured in S. We wish to find the frequency f' and wavelength λ' of the light as measured by an observer fixed in frame S$'$, which is moving with speed v toward S as shown in Figure 1.17a and b. In general, we expect f' to be greater than f if S$'$ approaches S because more wave crests are crossed per unit time, and we expect f' to be less than f if S$'$ recedes from S. In particular, consider the situation from the point of view of an observer fixed in S' as shown in Figure 1.18. This figure shows two successive wave fronts (color) emitted when the approaching source is at positions 1 and 2, respectively. If the time between the emission of these wavefronts as measured in S' is T', during this time front 1 will move a distance cT' from position 1. During this same time, the light source will advance a distance vT' to the left of position 1, and the distance between successive wave fronts will be measured in S$'$ to be

$$\lambda' = cT' - vT' \tag{1.9}$$

Because we wish to obtain a formula for f' (the frequency measured in S$'$) in terms of f (the frequency measured in S), we use $f' = c/\lambda'$ to obtain

$$f' = \frac{c}{(c - v)T'} \tag{1.10}$$

To eliminate T' in favor of T, note that T is the proper time; that is, T is the time between two events (the emission of successive wave fronts) that occur at the same place in S, and consequently,

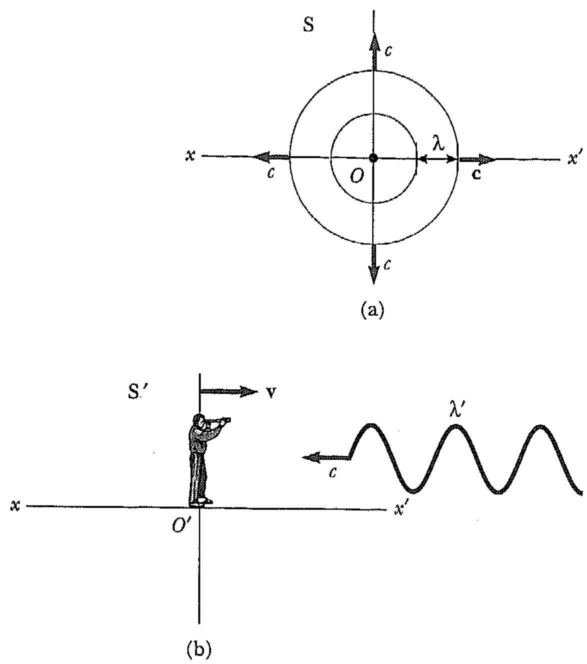


Figure 1.17 (a) A light source fixed in S emits wave crests separated in space by λ and moving outward at speed c as seen from S. (b) What wavelength λ' is measured by an observer at rest in S'? S' is a frame approaching S at speed v along $x - x'$.

$$T' = \frac{T}{\sqrt{1 - (v^2/c^2)}}$$

Substituting for T' in Equation 1.10 and using $f = 1/T$ gives

$$f' = \frac{\sqrt{1 - (v^2/c^2)}}{1 - (v/c)} f \tag{1.11}$$

or

$$f' = \frac{\sqrt{1 + (v/c)}}{\sqrt{1 - (v/c)}} f \tag{1.12}$$

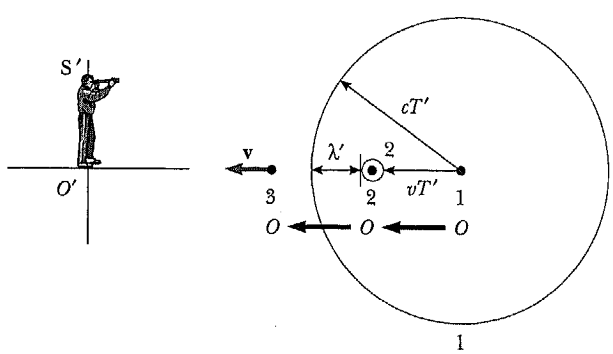


Figure 1.18 The view from S'. 1, 2, and 3 (in black) show three successive positions of O separated in time by T', the period of the light as measured from S'.

For clarity this expression is often written

$$f_{\text{obs}} = \frac{\sqrt{1 + (v/c)}}{\sqrt{1 - (v/c)}} f_{\text{source}} \tag{1.13}$$

where f_{obs} is the frequency measured by an observer approaching a light source, and f_{source} is the frequency as measured in the source's rest frame.

Equation 1.13 is the relativistic Doppler shift formula, which, unlike the Doppler formula for sound, depends only on the relative speed v of the source and observer and holds for relative speeds as large as c. Equation 1.13 agrees with physical intuition in predicting f_{obs} to be greater than f_{source} for an approaching emitter and receiver. The expression for the case of a receding source is obtained by replacing v with $-v$ in Equation 1.13.

Although Christian Johann Doppler's name is most frequently associated with the effect in sound, he originally developed his ideas in an effort to understand the shift in frequency or wavelength of the light emitted by moving atoms and astronomical objects. The most spectacular and dramatic use of the Doppler effect has occurred in just this area in explaining the famous red shift of absorption lines (wavelengths) observed for most galaxies. (A galaxy is a cluster of millions of stars.) The term *red shift* refers to the shift of known absorption lines toward longer wavelengths, that is, toward the red end of the visible spectrum. For example, lines normally found in the extreme violet region for a galaxy at rest with respect to the Earth are shifted about 100 nm toward the red end of the spectrum for distant galaxies—indicating that these distant galaxies are rapidly *receding* from us. The American astronomer Edwin Hubble used this technique to confirm that most galaxies are moving away from us and that the Universe is *expanding*. (For more about the expanding Universe see Chapter 15, Section 12.)

EXAMPLE 1.5 Determining the Speed of Recession of the Galaxy Hydra

The light emitted by a galaxy contains a continuous distribution of wavelengths because the galaxy is composed of millions of stars and other thermal emitters. However, some narrow gaps occur in the continuous spectrum where the radiation has been strongly absorbed by cooler gases in the galaxy (Fig. 15.19). In particular, a cloud of ionized calcium atoms produces very strong absorption at 394 nm for a galaxy at rest with respect to the Earth. For the galaxy Hydra, which is 2 billion lightyears away, this absorption is shifted to 475 nm. How fast is Hydra moving away from the Earth?

Solution For an approaching source and observer, $f_{\text{obs}} > f_{\text{source}}$ and $\lambda_{\text{obs}} < \lambda_{\text{source}}$ because $f_{\text{obs}}\lambda_{\text{obs}} = c = f_{\text{source}}\lambda_{\text{source}}$. In the case of Hydra $\lambda_{\text{obs}} > \lambda_{\text{source}}$, so Hydra must be receding and we must use

$$f_{\text{obs}} = \frac{\sqrt{1 - (v/c)}}{\sqrt{1 + (v/c)}} f_{\text{source}}$$

Substituting $f_{\text{obs}} = c/\lambda_{\text{obs}}$ and $f_{\text{source}} = c/\lambda_{\text{source}}$ into this equation gives

$$\lambda_{\text{obs}} = \frac{\sqrt{1 + (v/c)}}{\sqrt{1 - (v/c)}} \lambda_{\text{source}}$$

Finally, solving for v/c, we find

$$\frac{v}{c} = \frac{\lambda_{\text{obs}}^2 - \lambda_{\text{source}}^2}{\lambda_{\text{obs}}^2 + \lambda_{\text{source}}^2}$$

or

$$\frac{v}{c} = \frac{(475\text{ nm})^2 - (394\text{ nm})^2}{(475\text{ nm})^2 + (394\text{ nm})^2} = 0.185$$

Therefore, Hydra is receding from us at $v = 0.185c = 5.54 \times 10^7$ m/s.

1.6 THE LORENTZ TRANSFORMATION

We have seen that the Galilean transformation is not valid when v approaches the speed of light. In this section, we shall derive the correct transformation equations that apply for all speeds in the range of $0 \leq v < c$. This transformation, known as the Lorentz transformation, was developed by Hendrik A. Lorentz (1853–1928) in 1890 in connection with electromagnetism. However, its real significance in a physical theory transcending electromagnetism was first recognized by Einstein.

The Lorentz transformation is a set of formulas that relates the space and time coordinates of two inertial observers moving with a relative speed v. We have already seen two examples of the Lorentz transformation in the time dilation and length contraction formulas. The Lorentz transformation equations provide a more formal, concise, and almost mechanical method of solution that is well suited to further derivations such as the transformation of velocities between moving frames.

We start our derivation of the Lorentz transformation by noting that a reasonable guess about the form of these equations (based on physical intuition) can greatly reduce the algebraic complexity of the derivation. For simplicity, consider the standard frames, S and S′, with S′ moving at a speed v along the $+x$ direction (see Fig. 1.2). A reasonable guess about the dependence of x' on x and t is

$$x' = G(x - vt) \tag{1.14}$$

where G is a dimensionless factor that does not depend on x or t but is some function of v/c such that G is 1 in the limit as v/c approaches 0. The form of Equation 1.14 is suggested by the form of the Galilean transformation, $x' = x - vt$, which we know is correct for low speeds or in the limit as v/c approaches zero. The fact that Equation 1.14 is linear in x and t is also important because we require a single event in S (specified by x_1, t_1) to correspond to a single event in S′ (specified by x_1', t_1'). Assuming that Equation 1.14 is correct, we can write the *inverse Lorentz transformation* for x in terms of x' and t' as

$$x = G(x' + vt') \tag{1.15}$$

This follows from Einstein's postulate of relativity, which requires the laws of physics to have the same form in both S and S′ and where the sign of v has been changed to take into account the difference in direction of motion of the two frames.

Consider a rocket moving with a speed v along the xx' axes as in Figure 1.19. The frame of the rocket S′ is indicated with the coordinates (x', y', z', t'), and an observer in S uses coordinates (x, y, z, t). A flashbulb mounted on the rocket emits a pulse of light at the instant that the origins of the two reference frames coincide.

At the instant the flashbulb goes off and the two origins coincide, we define $t = t' = 0$. The light signal travels as a spherical wave, where the origin of the wavefront is the fixed point O where the flash originated. At some later time, a point such as P on the spherical wavefront is at a distance r from O and a distance r' from O', as shown schematically in Figure 1.19b. According to Einstein's second postulate, the speed of light should be c for both observers. Hence, the distance to the point P on the wavefront as measured by an observer

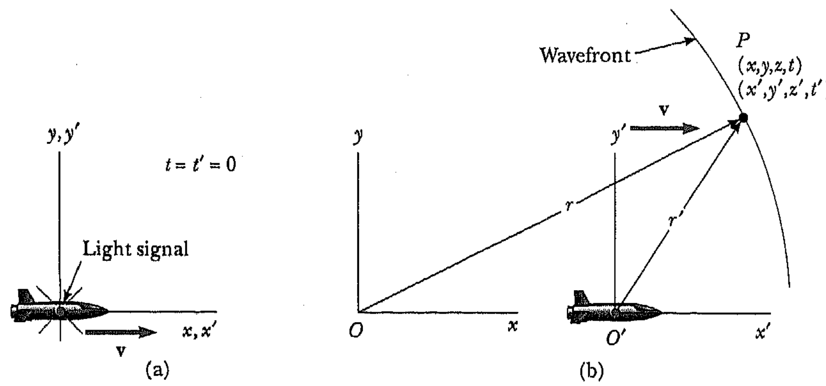


Figure 1.19 A rocket moves with a speed v along the xx' axes. (a) A pulse of light is sent out from the rocket at $t = t' = 0$ when the two systems coincide. (b) Coordinates of some point P on an expanding spherical wavefront as measured by observers in both inertial systems. (This figure is entirely schematic, and you should not be misled by the geometry.)

in S is given by $r = ct$, while the distance to the point P as measured by an observer in S' is given by $r' = ct'$. That is,

$$r = ct \tag{1.16}$$

$$r' = ct' \tag{1.17}$$

If we accept Einstein's second postulate we must require that the times t and t' taken for the light to reach P be *different.* This is in contrast to the Galilean transformation, where $t = t'$.

Because the motion of S' is along the xx' axes, it follows that the y and z coordinates measured in the two frames are always equal. That is, they are unaffected by the motion along x, and therefore $y = y'$ and $z = z'$.

To simplify matters consider the points x, x' along the x axes at which the spherical light wave crosses the horizontal axes. Equations 1.16 and 1.17 become $x = ct$ and $x' = ct'$. Substituting these values into Equations 1.14 and 1.15, we obtain

$$ct = G(ct' + vt') \tag{1.18}$$

$$ct' = G(ct - vt) \tag{1.19}$$

Solving Equation 1.19 for t' and substituting this value into Equation 1.18 gives

$$ct = \frac{G^2}{c}(c + v)(c - v)t \tag{1.20}$$

or

$$G \equiv \gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}} \tag{1.21}$$

The direct transformation is thus $x' = \gamma(x - vt)$, and the inverse transformation is $x = \gamma(x' + vt')$. To get the time transformation (t' as a function of t and x), substitute $x' = \gamma(x - vt)$ into $x = \gamma(x' + vt')$ to eliminate x':

$$x = \gamma[\gamma(x - vt) + vt'] \tag{1.22}$$

or

$$t' = \gamma\left[t + \frac{x}{v}\left(\frac{1}{\gamma^2} - 1\right)\right] \tag{1.23}$$

Because $1/\gamma^2 - 1 = -v^2/c^2$,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{1.24}$$

In summary, the complete transformations between an event found to occur at (x, y, z, t) in S and (x', y', z', t') in S′ are

Lorentz transformation for S → S′

$$x' = \gamma(x - vt) \tag{1.25}$$

$$y' = y \tag{1.26}$$

$$z' = z \tag{1.27}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{1.28}$$

$$\text{where} \quad \gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}} \tag{1.29}$$

If we wish to transform coordinates in the S′ frame to coordinates in the S frame, we simply replace v by $-v$ and interchange the primed and unprimed coordinates in Equations 1.25 through 1.29. The resulting transformation is given by

Inverse Lorentz transformation for S′ → S

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - (v^2/c^2)}} = \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \frac{t' + (v/c^2)x'}{\sqrt{1 - (v^2/c^2)}} = \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned} \tag{1.30}$$

In the Lorentz transformation, t depends on both t' and x'. Likewise, t' depends on both t and x. This is unlike the case of the Galilean transformation, in which $t = t'$.

When $v \ll c$, the Lorentz transformation should reduce to the Galilean transformation. To check this, note that as $v \to 0$, $v/c < 1$ and $v^2/c^2 \ll 1$, so that Equations 1.25–1.28 reduce in this limit to the Galilean coordinate transformation equations, given by

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

EXAMPLE 1.6 Time Dilation Is Contained in the Lorentz Transformation

Show that the phenomenon of time dilation is contained in the Lorentz transformation. A light located at (x_0, y_0, z_0) is turned abruptly on at t_1 and off at t_2 in frame S. (a) For what time interval is the light measured to be on in frame S′? (See Fig. 1.2 for a picture of the two standard frames.) (b) What is the distance between where the light is turned on and off as measured by S′?

Solution (a) The two events, the light turning on and the light turning off, are measured to occur in the two frames as follows:

	Event 1 (light on)	Event 2 (light off)
Frame S	$x_0,\ t_1$	$x_0,\ t_2$
Frame S′	$x_1' = \gamma(x_0 - vt_1)$	$x_2' = \gamma(x_0 - vt_2)$
	$t_1' = \gamma\left(t_1 - \frac{vx_0}{c^2}\right)$	$t_2' = \gamma\left(t_2 - \frac{vx_0}{c^2}\right)$

Note that the y and z coordinates are not affected because the motion of S′ is along x. As measured by S′, the light is on for a time interval

$$t_2' - t_1' = \gamma\left(t_2 - \frac{vx_0}{c^2}\right) - \gamma\left(t_1 - \frac{vx_0}{c^2}\right) = \gamma(t_2 - t_1)$$

Because $\gamma > 1$ and $(t_2 - t_1)$ is the proper time, it follows that $(t_2' - t_1') > (t_2 - t_1)$, and we have regained our previous result for time dilation, Equation 1.7.

(b) Although event 1 and event 2 occur at the same place in S, they are measured to occur at a separation of $x_2' - x_1'$ in S′ where

$$x_2' - x_1' = (\gamma x_0 - \gamma vt_2) - (\gamma x_0 - \gamma vt_1) = \gamma v(t_1 - t_2)$$

This result is reasonable because it reduces to

$$v(t_1 - t_2) \text{ for } v/c \ll 1.$$

Can you explain why $x_2' - x_1'$ is negative?

Exercise 2 Use the Lorentz transformation to derive the expression for length contraction. Note that the length of a moving object is determined by measuring the positions of both ends simultaneously.

Lorentz Velocity Transformation

Let us now derive the Lorentz velocity transformation, which is the relativistic counterpart of the Galilean velocity transformation. Suppose that an object is observed in the S′ frame with an instantaneous speed u_x' measured in S′ given by

$$u_x' = \frac{dx'}{dt'} \tag{1.31}$$

Using Equations 1.25 and 1.28, we have

$$dx' = \frac{dx - v\,dt}{\sqrt{1 - (v^2/c^2)}}$$

$$dt' = \frac{dt - (v/c^2)\,dx}{\sqrt{1 - (v^2/c^2)}}$$

Substituting these into Equation 1.31 gives

$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - (v/c^2)\,dx} = \frac{(dx/dt) - v}{1 - (v/c^2)(dx/dt)}$$

But dx/dt is just the velocity component u_x of the object measured in S, and so this expression becomes

Lorentz velocity transformation for S → S′

$$u_x' = \frac{u_x - v}{1 - (u_x v/c^2)} \tag{1.32}$$

Similarly, if the object has velocity components along y and z, the components in S′ are

$$u_y' = \frac{u_y}{\gamma[1 - (u_x v/c^2)]} \quad \text{and} \quad u_z' = \frac{u_z}{\gamma[1 - (u_x v/c^2)]} \tag{1.33}$$

When u_x and v are both much smaller than c (the nonrelativistic case), we see that the denominator of Equation 1.32 approaches unity, and so $u_x' \approx u_x - v$. This corresponds to the Galilean velocity transformations. In the other extreme, when $u_x = c$, Equation 1.32 becomes

$$u_x' = \frac{c - v}{1 - (cv/c^2)} = \frac{c[1 - (v/c)]}{1 - (v/c)} = c$$

From this result, we see that an object moving with a speed c relative to an observer in S also has a speed c relative to an observer in S′—*independent* of the relative motion of S and S′. Note that this conclusion is consistent with Einstein's second postulate, namely, that the speed of light must be c with respect to all inertial frames of reference. Furthermore, the speed of an object can never exceed c. That is, the speed of light is the "ultimate" speed. We return to this point later when we consider the energy of a particle.

To obtain u_x in terms of u_x', we replace v by $-v$ in Equation 1.32 and interchange the roles of u_x and u_x'. This gives

$$u_x = \frac{u_x' + v}{1 + (u_x' v/c^2)} \tag{1.34}$$

Inverse Lorentz velocity transformation for S′ → S

EXAMPLE 1.7 Relative Velocity of Spaceships

Two spaceships A and B are moving in *opposite* directions, as in Figure 1.20. An observer on Earth measures the speed of A to be $0.750c$ and the speed of B to be $0.850c$. Find the velocity of B with respect to A.

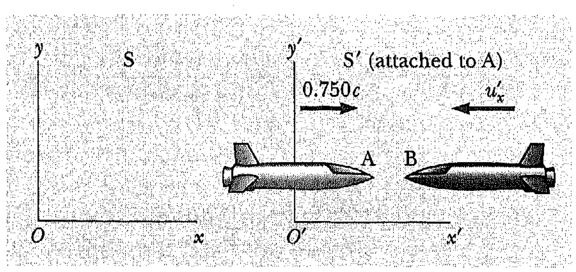


Figure 1.20 (Example 1.7) Two spaceships A and B move in *opposite* directions. The velocity of B relative to A is *less* than c and is obtained by using the relativistic velocity transformation.

Solution This problem can be solved by taking the S′ frame as being attached to spacecraft A, so that $v = 0.750c$ relative to an observer on Earth (the S frame). Spacecraft B can be considered as an object moving to the left with a velocity $u_x = -0.850c$ relative to the Earth observer. Hence, the velocity of B with respect to A can be obtained using Equation 1.32:

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.850c - 0.750c}{1 - \frac{(-0.850c)(0.750c)}{c^2}}$$
$$= -0.9771c$$

The negative sign for u_x' indicates that spaceship B is moving in the negative x direction as observed by A. Note that the result is less than c. That is, a body whose speed is less than c in one frame of reference must have a speed less than c in *any other* frame. If the Galilean velocity transformation were used in this example, we would find that $u_x' = u_x - v = -0.850c - 0.750c = -1.600c$, which is greater than c.

EXAMPLE 1.8 The Speeding Motorcycle

Imagine a motorcycle rider moving with a speed of $0.800c$ past a stationary observer, as shown in Figure 1.21. If the

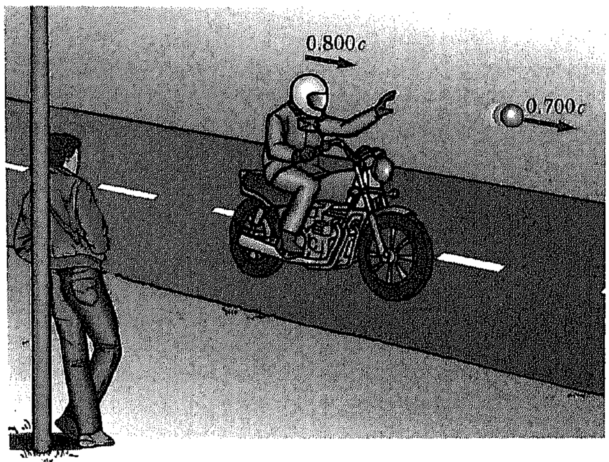


Figure 1.21 (Example 1.8) A motorcyclist moves past a stationary observer with a speed of $0.800c$ and throws a ball in the direction of motion with a speed of $0.700c$ relative to himself.

rider tosses a ball in the forward direction with a speed of $0.700c$ relative to himself, what is the speed of the ball as seen by the stationary observer?

Solution In this situation, the velocity of the motorcycle with respect to the stationary observer is $v = 0.800c$. The velocity of the ball in the frame of reference of the motorcyclist is $u'_x = 0.700c$. Therefore, the velocity, u_x, of the ball relative to the stationary observer is

$$u_x = \frac{u'_x + v}{1 + (u'_x v/c^2)}$$
$$= \frac{0.700c + 0.800c}{1 + [(0.700c)(0.800c)/c^2]} = 0.9615c$$

Exercise 3 Suppose that the motorcyclist moving with a speed $0.800c$ turns on a beam of light that moves away from him with a speed of c in the same direction as the moving motorcycle. What would the stationary observer measure for the speed of the beam of light?

Answer c.

EXAMPLE 1.9 Relativistic Leaders of the Pack!

Imagine two motorcycle gang leaders racing at relativistic speeds along perpendicular paths from the local pool hall as shown in Figure 1.22. How fast does pack leader Beta recede over Alpha's right shoulder as seen by Alpha?

Solution Figure 1.22 shows the situation as seen by a stationary police officer located in frame S, who observes

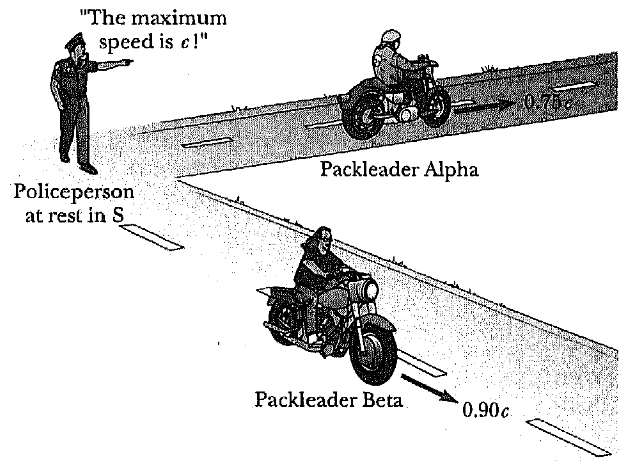


Figure 1.22 (Example 1.9) Two motorcycle pack leaders, Alpha and Beta, blaze past a stationary police officer. They are leading their respective gangs from the pool hall along perpendicular roads.

the following:

Pack leader Alpha	$u_x = 0.75c$	$u_y = 0$
Pack leader Beta	$u_x = 0$	$u_y = -0.90c$

To get Beta's speed of recession as seen by Alpha, we take S′ to move along with Alpha as shown in Figure 1.23, and we calculate u'_x and u'_y for Beta using Equations 1.32 and 1.33:

$$u'_x = \frac{u_x - v}{1 - (u_x v/c^2)} = \frac{0 - 0.75c}{1 - [(0)(0.75c)/c^2]} = -0.75c$$

$$u'_y = \frac{u_y}{\gamma[1 - (u_x v/c^2)]}$$
$$= \frac{\sqrt{1 - [(0.75c)^2/c^2]}(-0.90c)}{1 - [(0)(0.75c)/c^2]} = -0.60c$$

The speed of recession of Beta away from Alpha as observed by Alpha is then found to be less than c as required by relativity.

$$u' = \sqrt{(u'_x)^2 + (u'_y)^2} = \sqrt{(-0.75c)^2 + (-0.60c)^2} = 0.96c$$

Exercise 4 Calculate the classical speed of recession of Beta from Alpha using a Galilean transformation.

Answer $1.2c$

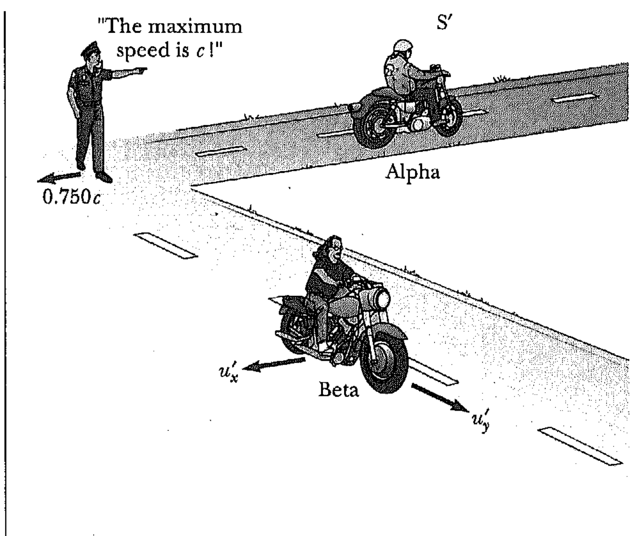


Figure 1.23 (Example 1.9) Pack leader Alpha's view of things.

1.7 RELATIVISTIC MOMENTUM AND THE RELATIVISTIC FORM OF NEWTON'S LAWS

Because Newton's laws are invariant under a Galilean transformation but we know that the correct relativistic transformation is the Lorentz transformation, we must correct or generalize Newton's laws to conform to the Lorentz transformation and the principles of relativity. These generalized definitions of Newton's laws of momentum and energy should reduce to the classical (nonrelativistic) definitions for $v \ll c$.

First, recall that the conservation of momentum states that when two bodies collide, the total momentum remains constant, assuming the bodies are isolated (that is, they interact only with each other). Suppose the collision is described in a reference frame S in which the momentum is conserved. If the velocities in a second reference frame S′ are calculated using the Lorentz transformation and the classical definition of momentum, $\mathbf{p} = m\mathbf{u}$, one finds that momentum *is not* conserved in the second reference frame. However, because the laws of physics are the same in all inertial frames, the momentum must be conserved in all systems. In view of this condition and assuming the Lorentz transformation is correct, we must modify the definition of momentum.

To see how the classical form $\mathbf{p} = m\mathbf{u}$ fails and to determine the correct relativistic definition of $\mathbf{p}$, consider the case of an inelastic collision between two particles of equal mass. Figure 1.24a shows such a collision for two identical particles approaching each other at speed v as observed in an inertial reference frame S. Using the classical form for momentum, $\mathbf{p} = m\mathbf{u}$, the observer in S finds momentum to be conserved as shown in Figure 1.24a. Suppose we now view things from an inertial frame S′ moving to the right with speed v with respect to S. In S′ the new speeds are v_1', v_2' and V' (see Fig. 1.24b). If we use the Lorentz velocity transformation

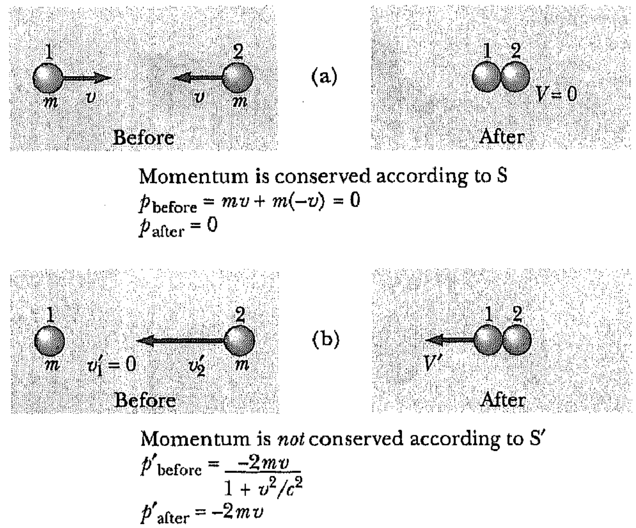


Figure 1.24 (a) An inelastic collision between two equal clay lumps as seen by an observer in frame S. (b) The same collision viewed from a frame S′ that is moving to the right with speed v with respect to S.

$$u_x' = \frac{u_x - v}{1 - (u_x v/c^2)} \tag{1.32}$$

to find v_1', v_2' and V', and the classical form for momentum, $\mathbf{p} = m\mathbf{u}$, will momentum be conserved according to the observer in S′? To answer this question we first calculate the velocities in S′ in terms of the given velocities in S.

$$v_1' = \frac{v_1 - v}{1 - (v_1 v/c^2)} = \frac{v - v}{1 - (v^2/c^2)} = 0$$

$$v_2' = \frac{v_2 - v}{1 - (v_2 v/c^2)} = \frac{-v - v}{1 - [(-v)(v)/c^2]} = \frac{-2v}{1 + (v^2/c^2)}$$

$$V' = \frac{V - v}{1 - (Vv/c^2)} = \frac{0 - v}{1 - [(0)v/c^2]} = -v$$

Checking for momentum conservation in S′, we have,

$$p'_{\text{before}} = mv_1' + mv_2' = m(0) + m\left[\frac{-2v}{1 + (v^2/c^2)}\right] = \frac{-2mv}{1 + (v^2/c^2)}$$

$$p'_{\text{after}} = 2mV' = -2mv$$

Thus, in the frame S′, the momentum before the collision is not equal to the momentum after the collision, and *momentum is not conserved.*

Conservation of linear momentum is a fundamental result, and it must be valid in all inertial frames. Because we have found that using the relativistically correct velocity transformation and the classical definition of momentum lead to the nonconservation of momentum, the only way to preserve conservation of momentum is to modify the definition of momentum. It can be shown (see

Example 1.15) that momentum is conserved in both S and S′, (and indeed in all inertial frames), if we redefine momentum as

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1-(u^2/c^2)}} \tag{1.35}$$

Definition of relativistic momentum

where **u** is the velocity of the particle and *m is the proper mass, that is, the mass measured by an observer at rest with respect to the mass.*[8] We use the symbol **u** for a particle velocity rather than **v**, which is used for the relative velocity of two reference frames. Note that when u is much less than c, the denominator of Equation 1.35 approached unity, so that **p** approaches $m\mathbf{u}$. Therefore, the relativistic equation for **p** reduces to the classical expression when u is small compared with c. Because it is simpler, Equation 1.35 is often written as $\mathbf{p} = \gamma m\mathbf{u}$, where $\gamma = 1/\sqrt{1-(u^2/c^2)}$. Note that this γ has the same functional form as the γ in the Lorentz transformation, but here γ contains u, the particle speed, and in the Lorentz transformation, γ contains v, the relative speed of the two frames.

The relativistic force **F** on a particle whose momentum is **p** is defined by the expression

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{dt} \tag{1.36}$$

where **p** is given by Equation 1.35. This expression is reasonable because it preserves classical mechanics in the limit of low velocities and requires the momentum of an isolated system ($\mathbf{F} = 0$) to be conserved relativistically as well as classically. It is left as a problem (Problem 29) to show that the acceleration a of a particle decreases under the action of a constant force, in which case $a \propto (1 - u^2/c^2)^{3/2}$. From this formula note that as the velocity approaches c, the acceleration caused by any finite force approaches zero. Hence, it is impossible to accelerate a particle from rest to a speed equal to or greater than c.

[8] In this book we shall always take m to be constant with respect to speed, and we call m the speed invariant mass, or proper mass. Some physicists refer to the mass in Equation 1.35 as the rest mass m_0 and call the term $m_0/\sqrt{1-(u^2/c^2)}$ the relativistic mass. Using this description, the relativistic mass is imagined to increase with increasing speed. We exclusively use the invariant mass m because we think it is a clearer concept and that the introduction of relativistic mass leads to no deeper physical understanding.

EXAMPLE 1.10 Momentum of an Electron

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of $0.750c$. Find its relativistic momentum and compare this with the momentum calculated from the classical expression.

Solution Using Equation 1.35 with $u = 0.750c$, we have

$$p = \frac{mu}{\sqrt{1-(u^2/c^2)}}$$

$$= \frac{(9.11 \times 10^{-31}\text{ kg})(0.750 \times 3.00 \times 10^8\text{ m/s})}{\sqrt{1-[(0.750c)^2/c^2]}}$$

$$= 3.10 \times 10^{-22}\text{ kg}\cdot\text{m/s}$$

The incorrect classical expression would give

$$\text{Momentum} = mu = 2.05 \times 10^{-22}\text{ kg}\cdot\text{m/s}$$

Hence, the correct relativistic result is 50% greater than the classical result!

EXAMPLE 1.11 An Application of the Relativistic Form of $\mathbf{F} = d\mathbf{p}/dt$: The Measurement of the Momentum of a High-Speed Charged Particle

Suppose a particle of mass m and charge q is injected with a relativistic velocity $\mathbf{u}$ into a region containing a magnetic field $\mathbf{B}$. The magnetic force $\mathbf{F}$ on the particle is given by $\mathbf{F} = q\mathbf{u} \times \mathbf{B}$. If $\mathbf{u}$ is perpendicular to $\mathbf{B}$, the force is radially inward, and the particle moves in a circle of radius R with $|\mathbf{u}|$ constant. From Equation 1.36 we have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{u})$$

Because the magnetic force is always perpendicular to the velocity, it does no work on the particle, and hence the speed, u, and γ are both constant with time. Thus, the magnitude of the force on the particle is

$$F = \gamma m \left|\frac{d\mathbf{u}}{dt}\right| \tag{1.37}$$

Substituting $F = quB$ and $|du/dt| = u^2/R$ (the usual definition of centripetal acceleration) into Equation 1.37, we can solve for $p = \gamma mu$. We find

$$p = \gamma mu = qBR \tag{1.38}$$

Equation 1.38 shows that the momentum of a relativistic particle of known charge q may be determined by measuring its radius of curvature R in a known magnetic field, $\mathbf{B}$. This technique is routinely used to determine the momentum of subatomic particles from photographs of their tracks through space.

1.8 RELATIVISTIC ENERGY

We have seen that the definition of momentum and the laws of motion required generalization to make them compatible with the principle of relativity. This implies that the relativistic form of the kinetic energy must also be modified.

To derive the relativistic form of the work-energy theorem, let us start with the definition of work done by a force F and make use of the definition of relativistic force, Equation 1.37. That is,

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx \tag{1.39}$$

where we have assumed that the force and motion are along the x axis. To perform this integration and find the work done on a particle or the relativistic kinetic energy as a function of the particle velocity u, we first evaluate dp/dt:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1 - (u^2/c^2)}} = \frac{m(du/dt)}{[1 - (u^2/c^2)]^{3/2}} \tag{1.40}$$

Substituting this expression for dp/dt and $dx = u\,dt$ into Equation 1.39 gives

$$W = \int_{x_1}^{x_2} \frac{m(du/dt)u\,dt}{[1 - (u^2/c^2)]^{3/2}} = m\int_0^u \frac{u\,du}{[1 - (u^2/c^2)]^{3/2}}$$

where we have assumed that the particle is accelerated from rest to some final velocity u. Evaluating the integral, we find that

$$W = \frac{mc^2}{\sqrt{1 - (u^2/c^2)}} - mc^2 \tag{1.41}$$

Recall that the work-energy theorem states that the work done by a force acting on a particle equals the change in kinetic energy of the particle. Because the initial kinetic energy is zero, we conclude that the work W is equivalent to the relativistic kinetic energy K, that is,

Relativistic kinetic energy

$$K = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} - mc^2 \tag{1.42}$$

This equation is routinely confirmed by experiments using high-energy particle accelerators. At low speeds, where $u/c \ll 1$, Equation 1.42 should reduce to the classical expression $K = \frac{1}{2}mu^2$. We can check this by using the binomial expansion $(1 - x^2)^{-1/2} \approx 1 + \frac{1}{2}x^2 + \cdots$ for $x \ll 1$, where the higher-order powers of x are ignored in the expansion. In our case, $x = u/c$, so that

$$\frac{1}{\sqrt{1-(u^2/c^2)}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

Substituting this into Equation 1.42 gives

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 = \frac{1}{2}mu^2$$

which agrees with the classical result. A graph comparing the relativistic and nonrelativistic expressions is given in Figure 1.25. Note that in the relativistic case, the particle speed never exceeds c, regardless of the kinetic energy. The two curves are in good agreement when $u \ll c$.

It is useful to write the relativistic kinetic energy in the form

$$K = \gamma mc^2 - mc^2 \tag{1.43}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

The constant term mc^2, which is independent of the speed, is called the **rest energy** of the particle. The term γmc^2, which depends on the particle speed,

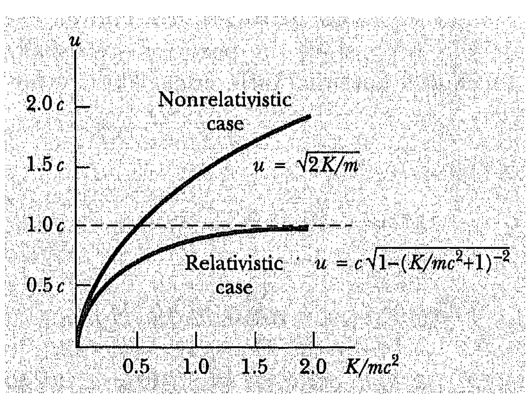


Figure 1.25 A graph comparing relativistic and nonrelativistic kinetic energy. The speeds are plotted versus energy. In the relativistic case, u is always *less* than c.

is therefore the sum of the kinetic and rest energies. We define γmc^2 to be the **total energy** E, that is,

Definition of total energy

$$E = \gamma mc^2 = K + mc^2 \tag{1.44}$$

The expression $E = \gamma mc^2$ is Einstein's famous mass-energy equivalence equation, which shows that mass is a measure of energy of all forms. Furthermore, this result shows that a small mass corresponds to an enormous amount of energy because c^2 is a very large number. This concept has revolutionized the field of nuclear physics and is treated in detail in the next section.

Mass-energy equivalence

In many situations, the momentum or energy of a particle is measured rather than its speed. It is therefore useful to have an expression relating the total energy E to the relativistic momentum p. This is accomplished by using the expression $E = \gamma mc^2$ and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate u (Problem 32). The result, after some algebra, is

Energy-momentum relation

$$E^2 = p^2c^2 + (mc^2)^2 \tag{1.45}$$

When the particle is at rest, $p = 0$, and so we see that $E = mc^2$. That is, the total energy equals the rest energy. For the case of particles that have zero mass, such as photons (massless, chargeless particles of light) and neutrinos (massless, chargeless particles associated with electron or beta-decay of a nucleus),[9] we set $m = 0$ in Equation 1.45, and we see that

$$E = pc \tag{1.46}$$

This equation is an *exact* expression relating energy and momentum for photons and neutrinos, which always travel at the speed of light.

Finally, note that because the mass m of a particle is independent of its motion, m must have the same value in all reference frames. On the other hand, the total energy and momentum of a particle depend on the reference frame in which they are measured, because they both depend on velocity. Because m is a constant, then according to Equation 1.45 the quantity $E^2 - p^2c^2$ must have the same value in all reference frames. That is, $E^2 - p^2c^2$ is *invariant* under a Lorentz transformation.

When dealing with electrons or other subatomic particles, it is convenient to express their energy in electron volts (eV), since the particles are usually given this energy by acceleration through a potential difference. The conversion factor is

$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$$

For example, the mass of an electron is 9.11×10^{-31} kg. Hence, the rest energy of the electron is

$$m_ec^2 = (9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 8.20 \times 10^{-14} \text{ J}$$

[9] Neutrino, from the Italian, means "little tiny neutral one." Following this practice, neutron should probably be neutrone (pronounced noo-trōn-eh) or "great big neutral one."

Converting this to electron volts, we have

$$m_ec^2 = (8.20 \times 10^{-14}\,\text{J})(1\,\text{eV}/1.60 \times 10^{-19}\,\text{J}) = 0.511\,\text{MeV}$$

where 1 MeV = 10^6 eV. Finally, note that because $mc^2 = 0.511$ MeV, the mass of the electron may be written $m = 0.511\,\text{MeV}/c^2$, accounting for the practice of measuring particle masses in units of MeV/c^2.

EXAMPLE 1.12 The Energy of a Speedy Electron

An electron has a speed $u = 0.850c$. Find its total energy and kinetic energy in electron volts.

Solution Using the fact that the rest energy of the electron is 0.511 MeV together with $E = \gamma mc^2$ gives

$$E = \frac{m_ec^2}{\sqrt{1-(u^2/c^2)}} = \frac{0.511\,\text{MeV}}{\sqrt{1-[(0.85c)^2/c^2]}}$$
$$= 1.90(0.511\,\text{MeV}) = 0.970\,\text{MeV}$$

The kinetic energy is obtained by subtracting the rest energy from the total energy:

$$K = E - m_ec^2 = 0.970\,\text{MeV} - 0.511\,\text{MeV} = 0.459\,\text{MeV}$$

EXAMPLE 1.13 The Energy of a Speedy Proton

The total energy of a proton is three times its rest energy.

(a) Find the proton's rest energy in electron volts.

$$\text{Rest energy} = m_pc^2$$
$$= (1.67 \times 10^{-27}\,\text{kg})(3.00 \times 10^8\,\text{m/s})^2$$
$$= (1.50 \times 10^{-10}\,\text{J})(1\,\text{eV}/1.60 \times 10^{-19}\,\text{J})$$
$$= 938\,\text{MeV}$$

(b) With what speed is the proton moving?

Because the total energy E is three times the rest energy, $E = \gamma mc^2$ gives

$$E = 3m_pc^2 = \frac{m_pc^2}{\sqrt{1-(u^2/c^2)}}$$

$$3 = \frac{1}{\sqrt{1-(u^2/c^2)}}$$

Solving for u gives

$$\left(1 - \frac{u^2}{c^2}\right) = \frac{1}{9} \quad \text{or} \quad \frac{u^2}{c^2} = \frac{8}{9}$$

$$u = \frac{\sqrt{8}}{3}c = 2.83 \times 10^8\,\text{m/s}$$

(c) Determine the kinetic energy of the proton in electron volts.

$$K = E - m_pc^2 = 3m_pc^2 - m_pc^2 = 2m_pc^2$$

Because $m_pc^2 = 938$ MeV, $K = 1876$ MeV.

(d) What is the proton's momentum?

We can use Equation 1.45 to calculate the momentum with $E = 3mc^2$:

$$E^2 = p^2c^2 + (m_pc^2)^2 = (3m_pc^2)^2$$
$$p^2c^2 = 9(m_pc^2)^2 - (m_pc^2)^2 = 8(m_pc^2)^2$$
$$p = \sqrt{8}\,\frac{m_pc^2}{c} = \sqrt{8}\,\frac{(938\,\text{MeV})}{c} = 2650\,\frac{\text{MeV}}{c}$$

Note that the unit of momentum is written MeV/c for convenience.

1.9 MASS AS A MEASURE OF ENERGY

The equation $E = \gamma mc^2$ as applied to a particle suggests that even when a particle is at rest ($\gamma = 1$) it still possesses enormous energy through its mass. The clearest experimental proof of the equivalence of mass and energy occurs in nuclear and elementary particle interactions in which both the conversion of mass into energy and the conversion of energy into mass take place. Because of this convertibility from the currency of mass into the currency of energy we can no longer accept the separate classical laws of the conservation of mass and the conservation of energy; we must instead speak of a single unified law, **the conservation of mass-energy.** Simply put, this law requires that *the sum of the mass-energy of a system of particles before interaction must equal the sum of the mass-energy of the system after interaction where the mass-energy of the ith particle is defined as*

Conservation of mass-energy

$$E_i = \frac{m_i c^2}{\sqrt{1 - (u_i^2/c^2)}}$$

To understand the conservation of mass-energy and to see how the new relativistic laws possess more symmetry and wider scope than the classical laws of momentum and energy conservation, we consider the simple inelastic collision treated earlier.

As one can see in Figure 1.24a, *classically* speaking momentum is conserved but kinetic energy is not because the total kinetic energy before collision equals mu^2 (we have replaced the v of Figure 1.24 with u) and the total kinetic energy after is zero. Now consider the same two colliding clay lumps using the relativistic mass-energy conservation law. If the mass of each lump is m, and the mass of the composite object is M, we must have:

$$E_{\text{before}} = E_{\text{after}}$$

$$\frac{mc^2}{\sqrt{1 - (u^2/c^2)}} + \frac{mc^2}{\sqrt{1 - (u^2/c^2)}} = Mc^2$$

or

$$M = \frac{2m}{\sqrt{1 - (u^2/c^2)}} \tag{1.47}$$

Because $\sqrt{1 - (u^2/c^2)} < 1$, the composite mass M is greater than the sum of the two individual masses! What's more, it is easy to show that the mass increase of the composite lump, $\Delta M = M - 2m$, is equal to the sum of the incident kinetic energies of the colliding lumps $(2K)$ divided by c^2:

$$\Delta M = \frac{2K}{c^2} = \frac{2}{c^2}\left(\frac{mc^2}{\sqrt{1 - (u^2/c^2)}} - mc^2\right) \tag{1.48}$$

Thus, we have an example of the conversion of kinetic energy to mass, and the satisfying result that in relativistic mechanics, kinetic energy is not lost in an inelastic collision but shows up as an increase in the mass of the final composite object. In fact the deeper symmetry of relativity theory shows that *both relativistic mass-energy and momentum are always conserved in a collision,* whereas classical methods show that momentum is conserved but kinetic energy is not unless the collision is perfectly elastic. Indeed, as we show in Example 1.15, relativistic momentum and energy are inextricably linked because momentum conservation in all inertial frames only holds if mass-energy conservation (Eq. 1.47) also holds.

EXAMPLE 1.14

(a) Calculate the mass increase for a completely inelastic head-on collision of two 5.0-kg balls each moving toward the other at 1000 miles per hour in opposite directions (the speed of a fast jet plane). (b) Explain why measurements on macroscopic objects have reinforced the *incorrect* beliefs that mass is conserved ($M = 2m$) and that kinetic energy is lost in an inelastic collision.

Solution (a) $u = 1000$ mi/h $= 450$ m/s

$$\frac{u}{c} = \frac{4.5 \times 10^2 \text{ m/s}}{3.0 \times 10^8 \text{ m/s}} = 1.5 \times 10^{-6}$$

Because $u^2/c^2 \ll 1$, substituting

$$\frac{1}{\sqrt{1 - (u^2/c^2)}} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$$

in Equation 1.48 gives

$$\begin{aligned}\Delta M &= 2m\left(\frac{1}{\sqrt{1 - (u^2/c^2)}} - 1\right)\\ &\approx 2m\left(1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right) \approx \frac{mu^2}{c^2}\\ &= (5.0 \text{ kg})(1.5 \times 10^{-6})^2 = 1.1 \times 10^{-11} \text{ kg}\end{aligned}$$

(b) Because the mass increase of 1.1×10^{-11} kg is an unmeasurably minute fraction of $2m$ (10 kg), it is quite natural to assume that the mass remains constant when macroscopic objects suffer an inelastic collision. On the other hand, the change in kinetic energy from mu^2 to 0 is so large (10^6 J) that it is readily measured to be lost in an inelastic collision of macroscopic objects.

Exercise 5 Prove that $\Delta M = 2\Delta K/c^2$ for a completely inelastic collision as stated.

EXAMPLE 1.15

Show that use of the relativistic definition of momentum

$$p = \frac{mu}{\sqrt{1 - (u^2/c^2)}}$$

leads to momentum conservation in both S and S′ for the inelastic collision shown in Figure 1.24.

Solution In frame S:

$$\begin{aligned}p_{\text{before}} &= \gamma mv + \gamma m(-v) = 0\\ p_{\text{after}} &= \gamma MV = (\gamma M)(0) = 0\end{aligned}$$

Hence, momentum is conserved in S. Note that we have used M as the mass of the two combined masses after the collision and allowed for the possibility in relativity that M is not necessarily equal to $2m$.

In frame S′:

$$\begin{aligned}p'_{\text{before}} &= \gamma m v'_1 + \gamma m v'_2 = \frac{(m)(0)}{\sqrt{1 - (0)^2/c^2}}\\ &\quad + \frac{m}{\{\sqrt{1 - [-2v/1 + (v^2/c^2)]^2}\}(1/c^2)}\left(\frac{-2v}{1 + v^2/c^2}\right)\end{aligned}$$

After some algebra, we find

$$\frac{m}{\{\sqrt{1 - [2v/1 + (v^2/c^2)]^2}\}(1/c^2)} = \frac{m(1 + v^2/c^2)}{(1 - v^2/c^2)}$$

and we obtain

$$\begin{aligned}p'_{\text{before}} &= \frac{m(1 + v^2/c^2)}{(1 - v^2/c^2)}\left(\frac{-2v}{1 + v^2/c^2}\right) = \frac{-2mv}{(1 - v^2/c^2)}\\ p'_{\text{after}} &= \gamma MV' = \frac{M(-v)}{\sqrt{1 - [(-v)^2/c^2]}} = \frac{-Mv}{\sqrt{1 - v^2/c^2}}\end{aligned}$$

To show that momentum is conserved in S′, we use the fact that M is not simply equal to $2m$ in relativity. As shown above, the combined mass, M, formed from the collision of two particles, each of mass m moving toward each other with speed v, is greater than $2m$. This occurs because of the equivalence of mass and energy, that is, the kinetic energy of the incident particles shows up in relativity theory but appears as another form of energy. Thus, from Equation 1.47 we have

$$M = \frac{2m}{\sqrt{1 - (v^2/c^2)}}$$

Substituting this result for M into p'_{after}, we obtain

$$\begin{aligned}p'_{\text{after}} &= \frac{2m}{\sqrt{1 - (v^2/c^2)}}\frac{-v}{\sqrt{1 - (v^2/c^2)}}\\ &= \frac{-2mv}{1 - (v^2/c^2)} = p'_{\text{before}}\end{aligned}$$

Hence, momentum is conserved in both S and S′, provided that we use the correct relativistic definition of momentum, $p = \gamma mu$, and invoke conservation of mass and energy.

The absence of observable effects of the conservation of mass and energy with inelastic collisions of macroscopic objects impels us to look for other areas to test this law, where particle velocities are higher, masses are more precisely known, and forces are stronger than electrical or mechanical forces. This leads us naturally to consider nuclear collisions and decays because nuclear masses can be measured very precisely with a mass spectrometer, nuclear forces are much stronger than electrical forces, and decay products are often produced with extremely high velocities.

Perhaps the most direct confirmation of the conservation of mass and energy occurs in the decay of a heavy radioactive nucleus at rest into several lighter particles that are emitted with large kinetic energies. For such a nucleus of mass

Fission

M undergoing *fission* into particles with masses M_1, M_2, and M_3 and having speeds u_1, u_2, and u_3, conservation of mass and energy requires

$$Mc^2 = \frac{M_1 c^2}{\sqrt{1 - (u_1^2/c^2)}} + \frac{M_2 c^2}{\sqrt{1 - (u_2^2/c^2)}} + \frac{M_3 c^2}{\sqrt{1 - (u_3^2/c^2)}} \tag{1.49}$$

Because the square roots are all less than one, $M > M_1 + M_2 + M_3$ and the loss of mass, $M - (M_1 + M_2 + M_3)$, appears as energy of motion of the products. This disintegration energy released per fission is often denoted by the symbol Q and can be written for our case as

$$Q = [M - (M_1 + M_2 + M_3)]c^2 = \Delta mc^2 \tag{1.50}$$

EXAMPLE 1.16 A Fission Reaction

An excited $^{236}_{92}\text{U}$ nucleus decays at rest into $^{90}_{37}\text{Rb}$, $^{143}_{55}\text{Cs}$, and several neutrons, ^1_0n. (a) By conserving charge and the total number of protons and neutrons write a balanced reaction equation and determine the number of neutrons produced. (b) Calculate by how much the combined "offspring" mass is less than the "parent" mass. (c) Calculate the energy released per fission. (d) Calculate the energy released in kilowatt hours when 1 kg of uranium undergoes fission in a power plant which is 40% efficient.

Solution In general an element is represented by the symbol ^A_ZX, where X represents the symbol for the element, A is the number of neutrons plus protons in the nucleus (mass number), and Z is the number of protons in the nucleus (atomic number). Conserving charge and number of nucleons gives:

$$^{236}_{92}\text{U} \longrightarrow {}^{90}_{37}\text{Rb} + {}^{143}_{55}\text{Cs} + 3\,{}^1_0\text{n}$$

So three neutrons are produced per fission.

(b) The masses of the decay particles are given in Appendix B in terms of atomic mass units, u, where 1 u = 1.660×10^{-27} = 931.5 MeV/c^2.

$$\begin{aligned}\Delta m &= M_{\text{U}} - (M_{\text{Rb}} + M_{\text{Cs}} + 3m_{\text{n}}) = 236.045563\text{ u} \\ &\quad - 89.914811\text{ u} + 142.927220\text{ u} + (3)(1.008665)\text{ u} \\ &= 0.177537\text{ u} = 2.9471 \times 10^{-28}\text{ kg}\end{aligned}$$

Therefore the reaction products have a combined mass that is about 3.0×10^{-28} kg less than the initial uranium mass.

(c) The energy given off per fission event is just Δmc^2. This is most easily calculated if Δm is first converted to mass units of MeV/c^2. Because 1 u = 931.5 MeV/c^2,

$$\Delta m = (0.177537\text{ u})(931.5\text{ MeV}/c^2) = 165.4\text{ MeV}/c^2$$

$$Q = \Delta mc^2 = 165.4\frac{\text{MeV}}{c^2}c^2 = 165.4\text{ MeV}$$

(d) To find the energy released by the fission of 1 kg of uranium we need to calculate the number of nuclei, N, contained in 1 kg of ^{236}U.

$$\begin{aligned}N &= \frac{(6.02 \times 10^{23}\text{ nuclei/mol})}{(236\text{ g/mol})}(1000\text{ g}) \\ &= 2.55 \times 10^{24}\text{ nuclei}\end{aligned}$$

The total energy produced, E, is

$$\begin{aligned}E &= (\text{efficiency})NQ \\ &= (0.40)(2.55 \times 10^{24}\text{ nuclei})(165\text{ MeV/nucleus}) \\ &= 1.68 \times 10^{26}\text{ MeV} \\ &= (1.68 \times 10^{26}\text{ MeV})(4.45 \times 10^{-20}\text{ kWh/MeV}) \\ &= 7.48 \times 10^6\text{ kWh}\end{aligned}$$

Exercise 6 How long will this amount of energy keep a 100-W lightbulb burning?

Answer ≈ 8500 years.

Although the simplest case showing the release of nuclear energy is the decay of a heavy unstable element into several lighter elements, the most common case is the one in which the mass of a composite particle is *less than* the sum of the particle masses composing it. By examining Appendix B, you can see that the mass of any nucleus is less than the sum of its component neutrons and protons by an amount Δm. This occurs because the nuclei are stable, bound systems of neutrons and protons (bound by strong attractive nuclear forces), and in order to disassociate them into separate nucleons an amount of energy Δmc^2 must be supplied to the nucleus. This energy or work required to pull a bound system apart, leaving its component parts free of attractive forces and at rest, is called the binding energy, *BE*. Thus, we describe the mass and energy of a bound system by the equation

$$Mc^2 + BE = \sum_{i=1}^{n} m_i c^2 \tag{1.51}$$

where M is the bound system mass, the m_i's are the free component particle masses, and n is the number of component particles. Two general comments are needed about Equation 1.51. First, it applies quite generally to any type of system bound by attractive forces, whether gravitational, electrical (chemical) or nuclear. For example, the mass of a water molecule is less than the combined mass of two free hydrogen atoms and a free oxygen atom, although the mass difference cannot be directly measured in this case. (The mass difference can be measured in the nuclear case because the forces and the binding energy are so much greater.) Second, Equation 1.51 shows the possibility of liberating huge quantities of energy, *BE*, if one reads the equation from right to left; that is, one collides nuclear particles with enough energy to overcome proton repulsion and fuse the particles into new elements with less mass. Such a process is called *fusion*, one example of which is a reaction in which two deuterium nuclei combine to form a helium nucleus, releasing 23.9 MeV per fusion. We can write this reaction schematically as follows:

Fusion

$$^2_1\text{H} + {}^2_1\text{H} \longrightarrow {}^4_2\text{He} + 23.9\ \text{MeV}$$

EXAMPLE 1.17

(a) How much lighter is a molecule of water than two hydrogen atoms and an oxygen atom? The binding energy of water is about 3 eV. (b) Find the fractional loss of mass per gram of water formed. (c) Find the total energy released (mainly as heat and light) when one gram of water is formed.

Solution (a) Equation 1.51 may be solved for the mass difference as follows:

$$\Delta m = (m_\text{H} + m_\text{H} + m_\text{O}) - M_{\text{H}_2\text{O}} = \frac{BE}{c^2} = \frac{3\ \text{eV}}{c^2}$$

$$= \frac{(3.0\ \text{eV})(1.6 \times 10^{-19}\ \text{J/eV})}{(3.0 \times 10^8\ \text{m/s})^2} = 5.3 \times 10^{-36}\ \text{kg}$$

(b) To find the fractional loss of mass per molecule we divide Δm by the mass of a water molecule, $M_{\text{H}_2\text{O}} = 18u = 3.0 \times 10^{-26}$ kg:

$$\frac{\Delta m}{M_{\text{H}_2\text{O}}} = \frac{5.3 \times 10^{-36}\ \text{kg}}{3.0 \times 10^{-26}\ \text{kg}} = 1.8 \times 10^{-10}$$

Because the fractional loss of mass per molecule is the same as the fractional loss per gram of water formed, 1.8×10^{-10} g of mass would be lost for each gram of water formed. Because this is much too small a mass to be measured directly, this calculation shows that nonconservation of proper mass does not generally show up as a measurable effect in chemical reactions.

(c) The energy released when one gram of H_2O is formed is simply the change in mass when one gram of water is formed times c^2:

$$E = \Delta mc^2 = (1.8 \times 10^{-13}\ \text{kg})(3.0 \times 10^8\ \text{m/s})^2 \approx 16\ \text{kJ}$$

This energy change, as opposed to the decrease in mass, is easily measured, providing another case similar to Example 1.14 in which mass changes are minute but energy changes are easily measured.

1.10 CONSERVATION OF RELATIVISTIC MOMENTUM, MASS, AND ENERGY

So far we have considered only cases of the conservation of mass and energy. By far, however, the most common and strongest confirmation of relativity theory comes from the daily application of relativistic momentum and energy conservation to elementary particle interactions. Often the measurement of momentum (from the path curvature in a magnetic field—see Example 1.11) and kinetic energy (from the distance a particle travels in a known substance before coming to rest) are enough when combined with conservation of momentum, mass, and energy to determine fundamental particle properties like mass, charge, and mean lifetime.

EXAMPLE 1.18 Measuring the Mass of the π^+ Meson

The π^+ meson (also called the pion) is a subatomic particle responsible for the strong nuclear force between protons and neutrons. It is observed to decay at rest into a μ^+ meson (muon) and a neutrino, denoted ν. Because the neutrino has no charge or mass (talk about elusive!), it leaves no track in a bubble chamber. (A bubble chamber is a large chamber filled with liquid hydrogen that shows the tracks of charged particles as a series of tiny bubbles.) However, the track of the charged muon is visible as it loses kinetic energy and comes to rest (Fig. 1.26). If the mass of the muon is known to be 106 MeV/c^2, and the kinetic energy, K, of the muon is measured to be 4.6 MeV from its track length, find the mass of the π^+.

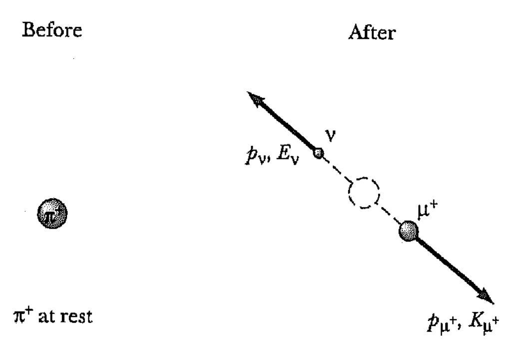


Figure 1.26 (Example 1.18) Decay of the pion at rest into a neutrino and a muon.

Solution Conserving energy gives

$$E_\pi = E_\mu + E_\nu$$

Because the pion is stopped when it decays, and the neutrino has zero mass,

$$m_\pi c^2 = \sqrt{(m_\mu c^2)^2 + (p_\mu{}^2 c^2)} + p_\nu c \qquad (1)$$

Conserving momentum in the decay yields $p_\mu = p_\nu$. Substituting the muon momentum for the neutrino momentum in Equation 1 gives the following expression for the rest energy of the pion in terms of the muon mass and momentum:

$$m_\pi c^2 = \sqrt{(m_\mu c^2)^2 + (p_\mu{}^2 c^2)} + p_\mu c \qquad (2)$$

Finally, to obtain p_μ from the measured value of the muon's kinetic energy, K_μ, we start with Equation 1.45, $E_\mu{}^2 = p_\mu{}^2 c^2 + (m_\mu c^2)^2$, and solve it for $p_\mu{}^2 c^2$:

$$\begin{aligned} p_\mu{}^2 c^2 &= E_\mu{}^2 - (m_\mu c^2)^2 = (K_\mu + m_\mu c^2)^2 - (m_\mu c^2)^2 \\ &= K_\mu{}^2 + 2K_\mu m_\mu c^2 \end{aligned}$$

Substituting this expression for $p_\mu{}^2 c^2$ into Equation (2) yields the desired expression for the pion mass in terms of the measured quantities:

$$\begin{aligned} m_\pi c^2 = &\sqrt{(m_\mu{}^2 c^4 + K_\mu{}^2 + 2K_\mu m_\mu c^2} \\ &+ \sqrt{K_\mu{}^2 + 2K_\mu m_\mu c^2} \end{aligned} \qquad (3)$$

Finally substituting $m_\mu c^2 = 106$ MeV and $K_\mu = 4.6$ MeV into Equation (3) gives

$$m_\pi c^2 = 111 \text{ MeV} + 31 \text{ MeV} \approx 1.4 \times 10^2 \text{ MeV}$$

Thus, the mass of the pion is

$$m_\pi = 140 \text{ MeV}/c^2$$

This result shows why this particle is called a meson; it has a *me*dium mass between the light electron (0.511 MeV/c^2) and the heavy proton (938 MeV/c^2).

SUMMARY

The two basic postulates of the **special theory of relativity** are as follows:

- The laws of physics must be the same for all observers moving at constant velocity with respect to one another.
- The speed of light must be the same for all inertial observers, independent of their relative motion.

To satisfy these postulates, the Galilean transformations must be replaced by the **Lorentz transformations** given by

$$x' = \gamma(x - vt) \tag{1.25}$$

$$y' = y \tag{1.26}$$

$$z' = z \tag{1.27}$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{1.28}$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}} \tag{1.29}$$

In these equations, it is assumed that the primed system moves with a speed v along the xx' axes.

The relativistic form of the **velocity transformation** is

$$u'_x = \frac{u_x - v}{1 - (u_x v/c^2)} \tag{1.32}$$

where u_x is the speed of an object as measured in the S frame and u'_x is its speed measured in the S′ frame.

Some of the consequences of the special theory of relativity are as follows:

- Clocks in motion relative to an observer appear to be slowed down by a factor γ. This is known as **time dilation.**
- Lengths of objects in motion appear to be contracted in the direction of motion by a factor of $1/\gamma$. This is known as **length contraction.**
- Events that are simultaneous for one observer are not simultaneous for another observer in motion relative to the first.

These three statements can be summarized by saying that duration, length, and simultaneity are not absolute concepts in relativity.

The relativistic expression for the **momentum** of a particle moving with a velocity **u** is

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - (u^2/c^2)}} = \gamma m\mathbf{u} \tag{1.35}$$

where γ is redefined in the following equation as

$$\gamma = \frac{1}{\sqrt{1 - (u^2/c^2)}}$$

The relativistic expression for the **kinetic energy** of a particle is

$$K = \gamma mc^2 - mc^2 \tag{1.42}$$

where mc^2 is called the **rest energy** of the particle.

The total energy E of a particle is related to the mass through the expression:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - (u^2/c^2)}} \tag{1.44}$$

Another useful expression relates the relativistic momentum to the total energy through the equation

$$E^2 = p^2c^2 + (mc^2)^2 \tag{1.45}$$

Finally, the law of the conservation of mass and energy states that the sum of the masses and energies of a system of particles before interaction must equal the sum of the masses and the energies of the particles after interaction where the energy of the ith particle is

$$E = \frac{m_i c^2}{\sqrt{1 - (u_i^2/c^2)}}$$

Application of the conservation of mass and energy to the specific cases of (1) the fission of a heavy nucleus at rest and (2) the fusion of several particles into a composite nucleus with less total mass allows us to define (1) the energy released per fission, Q, and (2) the binding energy of a composite system, BE.

SUGGESTIONS FOR FURTHER READING

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QUESTIONS

1. What two measurements will two observers in relative motion *always* agree on?
2. A spaceship in the shape of a sphere moves past an observer on Earth with a speed $0.5c$. What shape will the observer see as the spaceship moves past?
3. An astronaut moves away from Earth at a speed close to the speed of light. If an observer on Earth could make measurements of the astronaut's size and pulse rate, what changes (if any) would he or she measure? Would the astronaut measure any changes?
4. Two identically constructed clocks are synchronized. One is put in an eastward orbit around Earth while the other remains on Earth. Which clock runs slower? When the moving clock returns to Earth, will the two clocks still be synchronized?
5. Two lasers situated on a moving spacecraft are triggered simultaneously. An observer on the spacecraft claims to see the pulses of light simultaneously. What condition is necessary in order that another observer agrees that the two pulses are emitted simultaneously?
6. When we say that a moving clock runs slower than a stationary one, does this imply that there is something physically unusual about the moving clock?
7. When we speak of time dilation, do we mean that time passes more slowly in moving systems or that it simply appears to do so?
8. List some ways our day-to-day lives would change if the speed of light were only 50 m/s.
9. Give a physical argument to show that it is impossible to accelerate an object of mass m to the speed of light, even with a continuous force acting on it.
10. It is said that Einstein, in his teenage years, asked the question, "What would I see in a mirror if I carried it in my hands and ran at the speed of light?" How would you answer this question?
11. Because mass is a measure of energy, can we conclude that a compressed spring has more mass than the same spring when it is not compressed? On the basis of your answer, which has more mass, a spinning planet or an otherwise identical but nonspinning planet?
12. Suppose astronauts were paid according to the time spent traveling in space. After a long voyage at a speed near that of light, a crew of astronauts return and open their pay envelopes. What will their reaction be?
13. What happens to the density of an object as its speed increases, as measured by an Earth observer?

PROBLEMS

1.2 The Principle of Relativity

1. In a laboratory frame of reference, an observer notes that Newton's second law is valid provided it is expressed as $\mathbf{F} = d\mathbf{p}/dt$. Show that it is also valid for an observer moving at a constant speed relative to the laboratory frame.
2. Show that Newton's second law is not valid in a reference frame moving past the laboratory frame of Problem 1 with a constant acceleration.
3. A 2000-kg car moving with a speed of 20 m/s collides with and sticks to a 1500-kg car at rest at a stop sign. Show that because momentum is conserved in the rest frame, momentum is also conserved in a reference frame moving with a speed of 10 m/s in the direction of the moving car.
4. A billiard ball of mass 0.3 kg moves with a speed of 5 m/s and collides elastically with a ball of mass 0.2 kg moving in the opposite direction with a speed of 3 m/s. Show that because momentum is conserved in the rest frame, it is also conserved in a frame of reference moving with a speed of 2 m/s in the direction of the second ball.

1.3 Michelson-Morley Experiment

5. An airplane flying upwind, downwind, and crosswind shows the main principle of the Michelson-Morley experiment. A plane capable of flying at speed c in still air is flying in a wind of speed v. Suppose the plane flies upwind a distance L and then returns downwind to its starting point. (a) Find the time needed to make the round trip and compare it with the time to fly crosswind a distance L and return. Before calculating these times, sketch the two situations. (b) Compute the time difference for the two trips if $L = 100$ mi, $c = 500$ mi/h, and $v = 100$ mi/h.

1.5 Consequences of Special Relativity

6. With what speed will a clock have to be moving in order to run at a rate that is one half the rate of a clock at rest?
7. How fast must a meter stick be moving if its length is observed to shrink to 0.5 m?
8. A clock on a moving spacecraft runs 1 s slower per day relative to an identical clock on Earth. What is the relative speed of the spacecraft? (*Hint:* For $v/c \ll 1$, note that $\gamma \approx 1 + v^2/2c^2$.)
9. A meter stick moving in a direction parallel to its length appears to be only 75 cm long to an observer. What is the speed of the meter stick relative to the observer?
10. A spacecraft moves at a speed of $0.900c$. If its length is L as measured by an observer on the spacecraft, what is the length measured by a ground observer?
11. The average lifetime of a pi meson in its own frame of

reference is 2.6×10^{-8} s. (This is the proper lifetime.) If the meson moves with a speed of $0.95c$, what is (a) its mean lifetime as measured by an observer on Earth and (b) the average distance it travels before decaying, as measured by an observer on Earth?

12. An atomic clock is placed in a jet airplane. The clock measures a time interval of 3600 s when the jet moves with a speed of 400 m/s. How much longer or shorter a time interval does an identical clock held by an observer on the ground measure? (*Hint:* For $v/c \ll 1$, $\gamma \approx 1 + v^2/2c^2$.)
13. An astronaut at rest on Earth has a heartbeat rate of 70 beats/min. What will this rate be when the astronaut is traveling in a spaceship at $0.90c$ as measured (a) by an observer also in the ship and (b) by an observer at rest on the Earth?
14. The muon is an unstable particle that spontaneously decays into an electron and two neutrinos. If the number of muons at $t = 0$ is N_0, the number at time t is given by $N = N_0 e^{-t/\tau}$, where τ is the mean lifetime, equal to 2.2 μs. Suppose the muons move at a speed of $0.95c$ and there are 5.0×10^4 muons at $t = 0$. (a) What is the observed lifetime of the muons? (b) How many muons remain after traveling a distance of 3.0 km?
15. A rod of length L_0 moves with a speed v along the horizontal direction. The rod makes an angle of θ_0 with respect to the x' axis. (a) Show that the length of the rod as measured by a stationary observer is given by $L = L_0[1 - (v^2/c^2) \cos^2 \theta_0]^{1/2}$. (b) Show that the angle that the rod makes with the x axis is given by the expression $\tan \theta = \gamma \tan \theta_0$. These results show that the rod is both contracted and rotated. (Take the lower end of the rod to be at the origin of the primed coordinate system.)
16. *The classical Doppler shift for light.* A light source recedes from an observer with a speed v that is small compared with c. (a) Show that in this case, Equation 1.13 reduces to

$$\frac{\Delta f}{f} \approx -\frac{v}{c}$$

(b) Also show that in this case

$$\frac{\Delta \lambda}{\lambda} \approx \frac{v}{c}$$

(*Hint:* Differentiate $\lambda f = c$ to show that $\Delta\lambda/\lambda = -\Delta f/f$) (c) Spectroscopic measurements of an absorption line normally found at $\lambda = 397$ nm reveal a red shift of 20 nm for light coming from a galaxy in Ursa Major. What is the recessional speed of this galaxy?

17. Calculate, for the judge, how fast you were going in miles per hour when you ran the red light because it appeared Doppler-shifted green to you. Take red light to have a wavelength of 650 nm and green to have a wavelength of 550 nm.
18. (a) How fast and in what direction must galaxy A be moving if an absorption line found at 550 nm (green) for a stationary galaxy is shifted to 450 nm (blue) for A? (b) How fast and in what direction is galaxy B moving if it shows the same line shifted to 700 nm (red)?
19. *Doppler radar.* An important practical application of the Doppler effect is the use of radar to determine the speed of a moving object. In this case the Doppler shift of the electromagnetic radar signal reflected from the moving object is directly proportional to the radial speed of the moving object with respect to the radar transmitter. If a police radar transmitter radiates at 10.0 GHz, calculate the frequency shift observed by the police for a car traveling at (a) 60.0 mi/h and (b) 70.0 mi/h. (c) What measurement precision in frequency is required to distinguish between the two cars? State your answer as a fraction $\Delta f/f$. How can such a high precision be obtained? (1.00 mi/h = 0.447 m/s).

1.6 The Lorentz Coordinate and Velocity Transformations

20. Two spaceships approach each other, each moving with the *same* speed as measured by an observer on the Earth. If their *relative* speed is $0.70c$, what is the speed of each spaceship?
21. An electron moves to the right with a speed of $0.90c$ relative to the laboratory frame. A proton moves to the right with a speed of $0.70c$ relative to the electron. Find the speed of the proton relative to the laboratory frame.
22. An observer on Earth observes two spacecraft moving in the *same* direction toward the Earth. Spacecraft A appears to have a speed of $0.50c$, and spacecraft B appears to have a speed of $0.80c$. What is the speed of spacecraft A measured by an observer in spacecraft B?
23. *Speed of light in a moving medium.* The motion of a medium such as water influences the speed of light. This effect was first observed by Fizeau in 1851. Consider a light beam passing through a horizontal column of water moving with a speed v. (a) Show that if the beam travels in the same direction as the flow of water, the speed of light measured in the laboratory frame is given by

$$u = \frac{c}{n}\left(\frac{1 + nv/c}{1 + v/nc}\right)$$

where n is the index of refraction of the water. (*Hint:* Use the velocity transformation relation, Equation 1.34, and note that the speed of light with respect to the moving frame is given by c/n.) (b) Show that for $v \ll c$, the expression above is in good agreement with Fizeau's experimental result

$$u \approx \frac{c}{n} + v - \frac{v}{n^2}$$

This proves that the Lorentz velocity transformation and not the Galilean velocity transformation is correct for light.

24. An observer in frame S sees lightning simultaneously strike two points 100 m apart. The first strike occurs at $x_1 = y_1 = z_1 = t_1 = 0$ and the second at $x_2 = 100$ m, $y_2 = z_2 = t_2 = 0$. (a) What are the coordinates of these two events in a frame S′ moving in the standard configuration at $0.70c$ relative to S? (b) How far apart are the events in S′? (c) Are the events simultaneous in S′? If not, what is the difference in time between the events, and which event occurs first?
25. As seen from Earth, two spaceships A and B are approaching along perpendicular directions. If A is observed by an Earth observer to have velocity $u_y = -0.90c$ and B to have a velocity $u_x = +0.90c$, find the speed of ship A as measured by the pilot of B.

1.7 Relativistic Momentum and the Relativistic Form of Newton's Laws

26. Calculate the momentum of a proton moving with a speed of (a) $0.010c$, (b) $0.50c$, (c) $0.90c$. (d) Convert the answers of (a)–(c) to MeV/c.
27. An electron has a momentum that is 90% larger than its classical momentum. (a) Find the speed of the electron. (b) How would your result change if the particle were a proton?
28. Consider the relativistic form of Newton's second law. Show that when **F** is parallel to **v**

$$F = m\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \frac{dv}{dt}$$

where m is the mass of an object and v is its speed.

29. A charged particle moves along a straight line in a uniform electric field E with a speed v. If the motion and the electric field are both in the x direction, (a) show that the magnitude of the acceleration of the charge q is given by

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

(b) Discuss the significance of the dependence of the acceleration on the speed. (c) If the particle starts from rest at $x = 0$ at $t = 0$, find the speed of the particle and its position after a time t has elapsed. Comment on the limiting values of v and x as $t \to \infty$.

30. Recall that the magnetic force on a charge q moving with velocity **v** in a magnetic field **B** is equal to $q\mathbf{v} \times \mathbf{B}$. If a charged particle moves in a circular orbit with a fixed speed v in the presence of a constant magnetic field, use Newton's second law to show that the frequency of its orbital motion is

$$f = \frac{qB}{2\pi m}\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

31. Show that the momentum of a particle having charge e moving in a circle of radius R in a magnetic field B is given by $p = 300BR$, where p is in MeV/c, B is in teslas, and R is in meters.

1.8 Relativistic Energy

32. Show that the energy-momentum relationship given by $E^2 = p^2c^2 + (mc^2)^2$ follows from the expressions $E = \gamma mc^2$ and $p = \gamma mu$.
33. A proton moves at a speed of $0.95c$. Calculate its (a) rest energy, (b) total energy, and (c) kinetic energy.
34. An electron has a kinetic energy five times greater than its rest energy. Find (a) its total energy and (b) its speed.
35. Find the speed of a particle whose total energy is 50% greater than its rest energy.
36. A proton in a high-energy accelerator is given a kinetic energy of 50 GeV. Determine the (a) momentum and (b) speed of the proton.
37. An electron has a speed of $0.75c$. Find the speed of a proton which has (a) the same kinetic energy as the electron and (b) the same momentum as the electron.
38. Protons in an accelerator at the Fermi National Laboratory near Chicago are accelerated to an energy of 400 times their rest energy. (a) What is the speed of these protons? (b) What is their kinetic energy in MeV?
39. *How long will the Sun shine, Nellie?* The Sun radiates about 4.0×10^{26} J of energy into space each second. (a) How much mass is released as radiation each second? (b) If the mass of the Sun is 2.0×10^{30} kg, how long can the Sun survive if the energy release continues at the present rate?
40. Electrons in projection television sets are accelerated through a total potential difference of 50 000 V. (a) Calculate the speed of the electrons using the relativistic form of kinetic energy assuming the electrons start from rest. (b) Calculate the speed of the electrons using the classical form of kinetic energy. (c) Is the difference in speed significant in the design of this set in your opinion?
41. As noted in Section 1.8, the quantity $E - p^2c^2$ is an *invariant* in relativity theory. This means that the quantity $E^2 - p^2c^2$ has the same value in all inertial frames even though E and p have different values in different frames. Show this explicitly by considering the following case. A particle of mass m is moving in the $+x$ direction with speed u and has momentum p and energy E in the frame S. (a) If S′ is moving at speed v in the standard way, find the momentum p' and energy E' observed in S′. (*Hint:* Use the Lorentz velocity transformation to find

p' and E') Is $E = E'$ and $p = p'$? (b) Show that $E^2 - p^2c^2$ is explicitly equal to $E'^2 - p'^2c^2$.

1.9 Mass as a Measure of Energy

42. A radium isotope decays to a radon isotope ^{222}Rn by emitting an α particle (a helium nucleus) according to the decay scheme $^{226}\text{Ra} \rightarrow {}^{222}\text{Rn} + {}^{4}\text{He}$. The masses of the atoms are 226.0254 (Ra), 222.0175 (Rn), and 4.0026 (He). How much energy is released as the result of this decay?
43. Consider the decay $^{55}_{24}\text{Cr} \rightarrow {}^{55}_{25}\text{Mn} + e^-$, where e^- is an electron. The ^{55}Cr nucleus has a mass of 54.9279 u, and the ^{55}Mn nucleus has a mass of 54.9244 u. (a) Calculate the mass difference in MeV. (b) What is the maximum kinetic energy of the emitted electron?
44. Calculate the binding energy in MeV per nucleon in the isotope $^{12}_{6}$C. Note that the mass of this isotope is exactly 12 u, and the masses of the proton and neutron are 1.007276 u and 1.008665 u, respectively.
45. The free neutron is known to decay into a proton, an electron, and an antineutrino $\bar{\nu}$ (of zero rest mass) according to

$$\text{n} \longrightarrow \text{p} + \text{e}^- + \bar{\nu}$$

This is called *beta decay*, and will be discussed further in Chapter 13. The decay products are measured to have a total kinetic energy of 0.781 MeV $\pm$ 0.005 MeV. Show that this observation is consistent with the excess energy predicted by the Einstein mass-energy relationship.

1.10 Conservation of Relativistic Momentum, Mass, and Energy

46. An electron having kinetic energy $K = 1.000$ MeV makes a head-on collision with a positron at rest. (A positron is an antimatter particle that has the same mass as the electron but opposite charge.) In the collision the two particles annihilate each other and are replaced by two γ rays of equal energy, each traveling at equal angles θ with the electron's direction of motion. (Gamma rays are massless particles of electromagnetic radiation having energy $E = pc$.) Find the energy E, momentum p, and angle of emission θ of the γ rays.
47. The K^0 meson is an uncharged member of the particle "zoo" that decays into two charged pions according to $\text{K}^0 \rightarrow \pi^+ + \pi^-$. The pions have opposite charges as indicated and the same mass, $m_\pi = 140$ MeV/c^2. Suppose that a K^0 at rest decays into two pions in a bubble chamber in which a magnetic field of 2.0 T is present (see Fig. P1.47). If the radius of curvature of the pions is 34.4 cm, find (a) the momenta and speeds of the pions and (b) the mass of the K^0 meson.

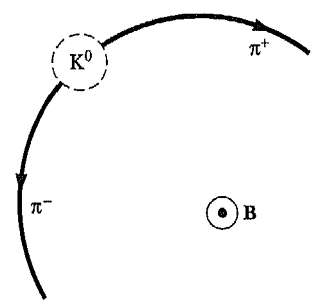


Figure P1.47 A sketch of the tracks made by the π^+ and π^- in the decay of the K^0 meson at rest. The pion motion is perpendicular to **B**. (**B** is directed out of the page.)

48. An unstable particle having a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocities of $0.987c$ and $-0.868c$. Find the rest masses of the fragments.

ADDITIONAL PROBLEMS

49. In 1962, when Scott Carpenter orbited Earth 22 times, the press stated that for each orbit he aged 2 millionths of a second less than if he had remained on Earth. (a) Assuming that he was 160 km above Earth in an eastbound circular orbit, determine the time difference between someone on Earth and the orbiting astronaut for the 22 orbits. You will need to use the approximation, $\sqrt{1 - x} \approx 1 - x/2$, for small x. (b) Did the press report accurate information? Explain.
50. The proper length of one spaceship is three times that of another. The two spaceships are traveling in the same direction and, while both are passing overhead, an Earth observer measures the two spaceships to have the same length. If the slower spaceship is moving with a speed of $0.35c$, determine the speed of the faster spaceship.
51. The pion has an average lifetime of 26.0 ns when at rest. For it to travel 10.0 m, how fast must it move?
52. If astronauts could travel at $v = 0.95c$, we on Earth would say it takes $(4.2/0.95) = 4.4$ years to reach Alpha Centauri, 4.2 lightyears away. The astronauts disagree. (a) How much time passes on the astronauts' clocks? (b) What distance to Alpha Centauri do the astronauts measure?
53. A spaceship moves away from Earth at a speed v and fires a shuttle craft in the forward direction at a speed v relative to the ship. The pilot of the shuttle craft launches

a probe at speed v relative to the shuttle craft. Determine (a) the speed of the shuttle craft relative to Earth, and (b) the speed of the probe relative to Earth.

54. In a nuclear power plant the fuel rods last three years before being replaced. If a 1.0-GW plant operates at 80% capacity for the three years, how much fuel is consumed?

55. An observer in a rocket moves toward a mirror at speed v relative to the reference frame labeled by S in Figure P1.55. The mirror is stationary with respect to S. A light pulse emitted by the rocket travels toward the mirror and is reflected back to the rocket. The front of the rocket is a distance d from the mirror (as measured by observers in S) at the moment the light pulse leaves the rocket. What is the total travel time of the pulse as measured by observers in (a) the S frame and (b) the front of the rocket?

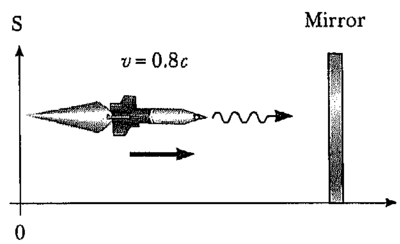


Figure P1.55

56. A physics professor on Earth gives an exam to her students who are on a spaceship traveling at speed v relative to Earth. The moment the ship passes the professor, she signals the start of the exam. If she wishes her students to have time T_0 (spaceship time) to complete the exam, show that she should wait a time (Earth time) of

$$T = T_0\sqrt{\frac{1 - v/c}{1 + v/c}}$$

before sending a light signal telling them to stop. (*Hint:* Remember that it takes some time for the second light signal to travel from the professor to the students.)

57. A yet-to-be-built spacecraft starts from Earth moving at constant speed to the yet-to-be-discovered planet Retah, which is 20 lighthours away from Earth. It takes 25 h (according to an Earth observer) for a spacecraft to reach this planet. Assuming that the clocks are synchronized at the beginning of the journey, compare the time elapsed in the spacecraft's frame for this one-way journey with the time elapsed as measured by an Earth-based clock.

58. Suppose our Sun is about to explode. In an effort to escape, we depart in a spaceship at $v = 0.80c$ and head toward the star Tau Ceti, 12 lightyears away. When we reach the midpoint of our journey from the Earth, we see our Sun explode and, unfortunately, at the same instant we see Tau Ceti explode as well. (a) In the spaceship's frame of reference, should we conclude that the two explosions occurred simultaneously? If not, which occurred first? (b) In a frame of reference in which the Sun and Tau Ceti are at rest, did they explode simultaneously? If not, which exploded first?

59. Two powerless rockets are on a collision course. The rockets are moving with speeds of $0.800c$ and $0.600c$ and are initially 2.52×10^{12} m apart as measured by Liz, an Earth observer, as shown in Figure P1.59. Both rockets are 50.0 m in length as measured by Liz. (a) What are their respective proper lengths? (b) What is the length of each rocket as measured by an observer in the other rocket? (c) According to Liz, how long before the rockets collide? (d) According to rocket 1, how long before they collide? (e) According to rocket 2, how long before they collide? (f) If both rocket crews are capable of total evacuation within 90 min (their own time), will there be any casualties?

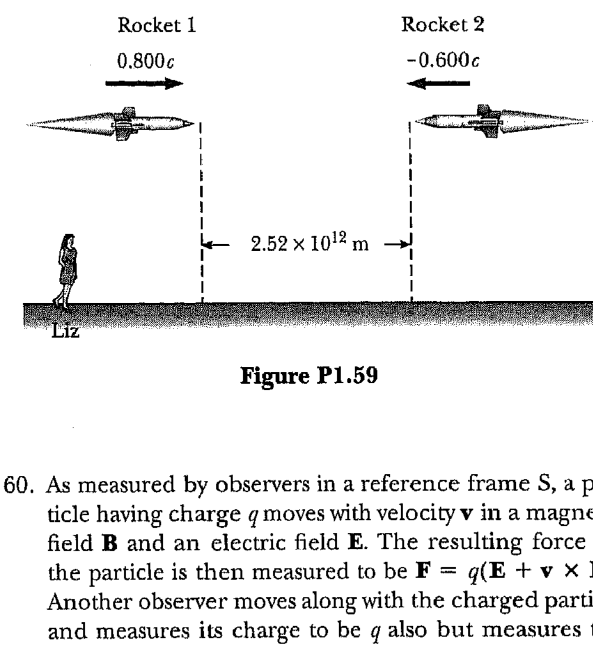
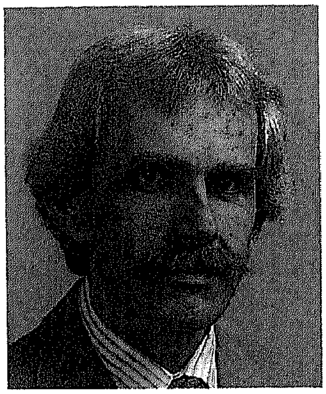


Figure P1.59

60. As measured by observers in a reference frame S, a particle having charge q moves with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$. The resulting force on the particle is then measured to be $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Another observer moves along with the charged particle and measures its charge to be q also but measures the electric field to be $\mathbf{E}'$. If both observers are to measure the same force, $\mathbf{F}$, show that $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$.

Essay

THE RENAISSANCE OF GENERAL RELATIVITY



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Despite its enormous influence on scientific thought in its early years, general relativity had become a sterile, formalistic subject by the late 1950s, cut off from the mainstream of physics. Yet by 1970, it had become one of the most active and exciting branches of physics. It took on new roles both as a theoretical tool of the astrophysicist and as a playground for the elementary-particle physicist. New experiments verified its predictions in unheard-of ways and to remarkable levels of precision. One of the most remarkable and important aspects of this renaissance was the degree to which experiment and observation motivated and complemented theoretical advances.

This was not always the case. In deriving general relativity during the final months of 1915, Einstein was not particularly motivated by a desire to account for observational results. Instead, he was driven by purely theoretical ideas of elegance and simplicity. His goal was to produce a theory of gravitation that incorporated both the special theory of relativity, which deals with physics in inertial frames, and the principle of equivalence, the proposal that physics in a frame falling freely in a gravitational field is in some sense equivalent to physics in an inertial frame.

Once he formulated the general theory, however, Einstein proposed three tests. One was an immediate success: the explanation of the anomalous advance in the perihelion of Mercury of 43 arcseconds per century, a problem that had bedeviled celestial mechanicians of the latter part of the 19th century. The next test, the deflection of light by the Sun, was such a success that it produced what today would be called a "media event." The third test, the gravitational red shift of light, remained unfulfilled until 1960, by which time it was no longer viewed as a true test of general relativity.

The turning point for general relativity came in the early 1960s, when discoveries of unusual astronomical objects such as quasars demonstrated that the theory has important applications in astrophysics. After 1960, the pace of research in general relativity and in an emerging field called "relativistic astrophysics" accelerated.

EINSTEIN'S THREE TESTS

Perihelion Shift The explanation of the anomalous perihelion shift of Mercury's orbit was an early triumph of general relativity. This had been an unsolved problem in celestial mechanics for over half a century, since the announcement by Urbain Le Verrier in 1859 that, after the perturbing effects of the planets on Mercury's orbit had been accounted for, there remained in the data an unexplained advance in the perihelion of Mercury. The modern value for this discrepancy is 42.98 arcseconds per cenury, and general relativity accounted for it in a natural way (Fig. E1). Radar measurements of the orbit of Mercury since 1966 have led to improved accuracy, so that the predicted relativistic advance agrees with the observations to about 0.1%.

Deflection of Light Einstein's second test concerns the deflection of light. According to general relativity, a light ray that passes the Sun at a distance d is deflected by an angle $\Delta\theta = 1.75''/d$, where d is measured in units of the solar radius, and the notation $''$ denotes seconds of arc (Fig. E2). Confirmation by the British astronomers Arthur S. Eddington and Crommelin of the bending of optical starlight observed during a total solar eclipse in the first months following World War I helped make Einstein a celebrity. However, those measurements had only 30% accuracy, and succeeding eclipse experiments weren't much better.

The development of long-baseline radio interferometry produced a method for greatly improved determinations of the deflection of light. Coupled with this techno-

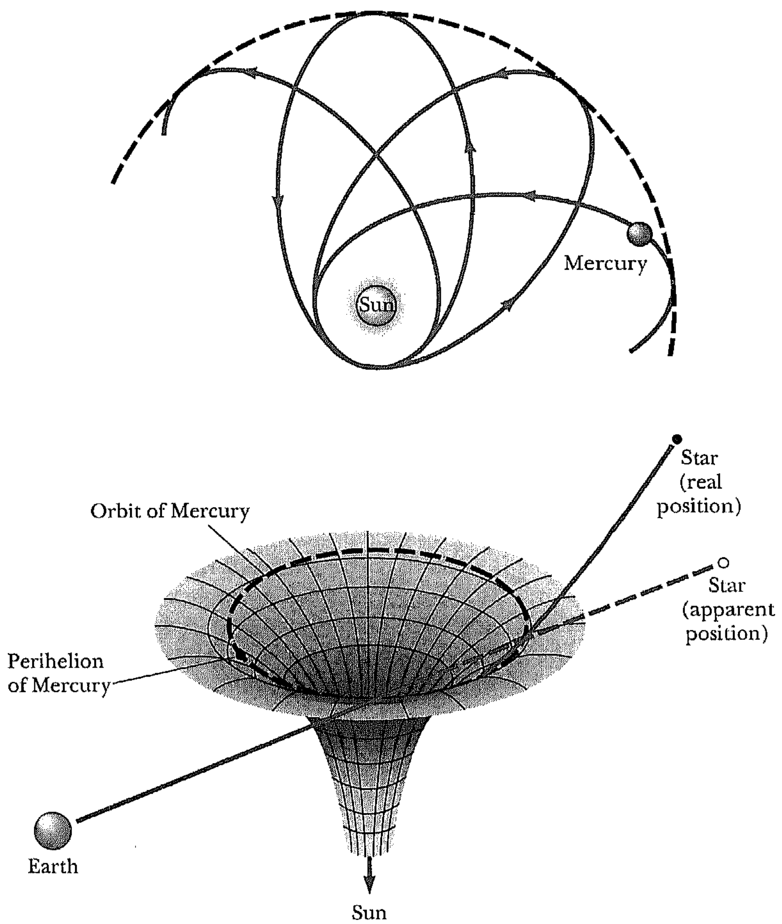

Figure E1 (a) Advance of the perihelion of Mercury. The elliptical orbit of Mercury about the Sun rotates very slowly relative to the system connected with the Sun. General relativity successfully explains this small effect, which predicts that the direction of the perihelion should change by only 43 arcseconds per century. (b) The diagram shows how a two-dimensional surface warped into three dimensions can change the direction of a "straight" line that is constrained to its surface; the warping of space is analogous, although with a greater number of dimensions to consider. The effect is similar to the golfer putting on a warped green. Though the ball is hit in a straight line, we see it appear to curve. (From Jay M. Pasachoff, *Astronomy: From The Earth to the Universe,* 4th ed., Philadelphia, Saunders College Publishing, 1991.)

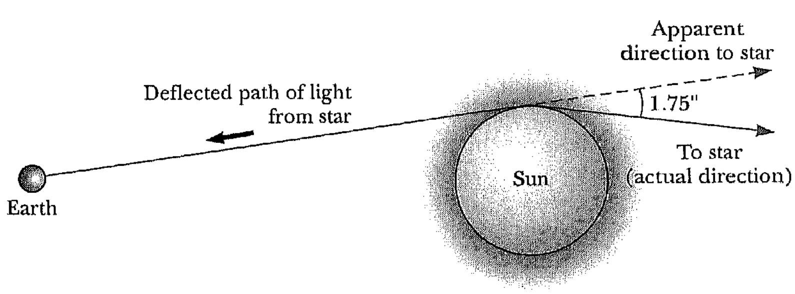

Figure E2 Deflection of starlight passing near the Sun. Because of this effect, the Sun and other remote objects can act as a *gravitational lens.* In his general relativity theory, Einstein calculated that starlight just grazing the Sun's surface should be deflected by an angle of 1.75."

logical advance is a series of heavenly coincidences: each year groups of strong quasars pass near the Sun (as seen from Earth). The idea is to measure the differential deflection of radio waves from one quasar relative to those from another as they pass near the Sun. A number of measurements of this kind occurred almost annually over the period from 1969 to 1975, yielding a confirmation of the predicted deflection to 1.5%. Recent measurements have improved this to 0.1%.

Gravitational Red Shift Another consequence of Einstein's insight is the gravitational red shift effect, which is a frequency shift between two identical clocks placed at different heights in a gravitational field. For small differences in height h between clocks, the shift in the frequency Δf is given by

$$\frac{\Delta f}{f} = \frac{gh}{c^2}$$

where g is the local gravitational acceleration and c is the speed of light. If the receiver clock is at a lower height than the emitter clock, the received signal is shifted to higher frequencies ("blue shift"); if the receiver is higher, the signal is shifted to lower frequencies ("red shift").

The most precise gravitational red shift experiment performed to date was a rocket experiment carried out in June 1976. A "hydrogen maser" atomic clock was flown on a Scout D rocket to an altitude of 10 000 km, and its frequency compared with a similar clock on the ground using radio signals. After the effects of the rocket's motion were taken into account, the observations confirmed the gravitational red shift to 0.02%.

APPLICATIONS

General relativity has passed every experimental test to which it has been put, and many alternative theories have fallen by the wayside. Most physicists now take the theory for granted and look to see how it can be used as a practical tool in physics and astronomy. For example, gravitational red shift must now be taken into account properly in order for the Global Positioning System (GPS) to perform its navigation function accurately.

Gravitational Radiation One of these new tools is gravitational radiation, a subject almost as old as general relativity. By 1916, Einstein had succeeded in showing that the field equations of general relativity admitted wave-like solutions analogous to those of electromagnetic theory. For example, a dumbbell rotating about an axis passing at right angles through its handle will emit gravitational waves that travel at the speed of light. They also carry energy away from the dumbbell, just as electromagnetic waves carry energy away from a light source.

In 1968, Joseph Weber initiated a program of gravitational-wave detection using massive aluminum bars as detectors. A passing gravitational wave acts as an oscillating gravitational force field that alternately compresses and extends the bar lengthwise. Currently a dozen laboratories around the world are engaged in building and improving on the basic "Weber bar" detector, striving to reduce noise from thermal, electrical, and environmental sources in order to detect the very weak oscillations produced by a gravitational wave. For a bar of one meter length, the challenge is to detect a variation in length smaller than 10^{-20} meters, or 10^{-5} of the radius of a proton. At the same time, several "laser-interferometric" gravitational-wave observatories are under construction in the United States and Europe. They are bouncing laser beams to monitor variations in length between mirrors spaced several kilometers apart. The observatories are scheduled to be operational around the year 2000.

Although gravitational radiation has not been detected directly, we know that it exists, through a remarkable system known as the binary pulsar. Discovered in 1974 by radio astronomers Russell Hulse and Joseph Taylor, it consists of a pulsar (which is a rapidly spinning neutron star) and a companion star in orbit around each other. Although the companion has not been seen directly, it is also believed to be a neutron star. The pulsar acts as an extremely stable clock, its pulse period of approximately 59 milliseconds drifting by only 0.25 ns/year. By measuring the arrival times of radio pulses at Earth, observers were able to determine the motion of the pulsar about its companion with amazing accuracy. For example, the accurate value for the orbital period is 27 906.980 895 s, and the orbital eccentricity is 0.617 131. Like a rotating dumbbell, an orbiting binary system should emit gravitational radiation and, in the process, lose some of its orbital energy. This energy loss will cause the pulsar and its companion to spiral in toward each other, and the orbital period to shorten. According to general relativity, the predicted decrease in the orbital period is 75.8 μs/year. The observed decrease rate is in agreement with the prediction to better than 0.5%. This confirms the existence of gravitational radiation and the general relativistic equations that describe it. Hulse and Taylor received the Nobel prize in 1993 for this discovery.

Black Holes The first glimmerings of the black hole idea date to the 18th century, in the writings of a British amateur astronomer, the Reverend John Michell. Reasoning on the basis of the corpuscular theory that light would be attracted by gravity, he noted that the speed of light emitted from the surface of a massive body would be reduced by the time the light was very far from the source. (Michell of course did not know special relativity.) It would therefore be possible for a body to be sufficiently massive and compact to prevent light from escaping from its surface.

Although the general relativistic solution for a nonrotating black hole was discovered by Karl Schwarzschild in 1916, and a calculation of gravitational collapse to a black hole state was performed by J. Robert Oppenheimer and Hartland Snyder in 1939, black hole physics didn't really begin until the middle 1960s, when astronomers confronted the problem of the energy output of the quasars and started to take black holes seriously.

A black hole is formed when a star has exhausted the thermonuclear fuel necessary to produce the heat and pressure that support it against gravity. The star begins to collapse, and if it is massive enough, it continues to collapse until its radius approaches a value called the gravitational radius, or Schwarzschild radius. In the nonrotating spherical case, this radius is $2GM/c^2$, where M is the mass of the star. For a body of one solar mass, the gravitational radius is about 3 km; for a body of the mass of Earth, it is about 9 mm. An observer sitting on the surface of the star sees the collapse continue to smaller and smaller radii, until both star and observer reach the origin $r = 0$, with consequences too horrible to describe. On the other hand, an observer at great distances observes the collapse to slow down as the radius approaches the gravitational radius, a result of the gravitational red shift of the light signals sent outward. The distant observer never sees any signals emitted by the falling observer once the latter is inside the gravitational radius. This is because any signal emitted inside can never escape the sphere bounded by the gravitational radius, called the "event horizon."

In 1974, Stephen Hawking discovered that the laws of quantum mechanics require a black hole to evaporate by the creation of particles with a thermal energy spectrum and to have an associated temperature and entropy. The temperature of a Schwarzschild black hole is $T = hc^3/8\pi k_B GM$, where h is Planck's constant and k_B is Boltzmann's constant. This discovery demonstrated a remarkable connection between gravity, thermodynamics, and quantum mechanics that helped renew the theoretical quest for a grand synthesis of all the fundamental interactions. On the other hand, for black holes of astronomical masses, the evaporation is completely negligible, because for a solar-mass black hole, $T \approx 10^{-6}$ K.

Although a great deal is known about black holes in theory, rather less is known about them observationally. There are several instances in which the evidence for the existence of black holes is impressive, but in all cases it is indirect. For instance, in the x-ray source

Cygnus XI, the source of the x-rays is believed to be a black hole with a mass larger than about six solar masses in orbit around a giant star. The x-rays are emitted by matter pulled from the surface of the companion star and sent into a spiralling orbit around the black hole. Recently, the Hubble Space Telescope detected convincing evidence for a black hole, billions of times the mass of the Sun, in the center of galaxy M87.

Cosmology Although Einstein in 1917 first used general relativity to calculate a model for the Universe, the subject was not considered a serious branch of physics until the 1960s, when astronomical observations lent credence to the idea that the Universe is expanding from a "Big Bang." In 1965 came the discovery of the cosmic background radiation by Arno Penzias and Robert Wilson. This radiation is the remains of the hot electromagnetic blackbody radiation that once dominated the Universe in its earlier phase, now cooled to 3 K by the subsequent expansion of the Universe. Next came calculations of the amount of helium synthesized from hydrogen by thermonuclear fusion in the very early Universe, around 1000 s after the Big Bang. The amount, approximately 25% by mass, was in agreement with the abundances of helium observed in stars and in interstellar space. This was an important confirmation of the hot Big-Bang picture because the amount of helium believed to be produced by fusion in the interiors of stars is woefully inadequate to explain the observed abundances.

Today, the general relativistic hot Big-Bang model of the Universe has broad acceptance, and cosmologists now focus their attention on more detailed issues, such as how galaxies and other large-scale structures formed out of the hot primordial soup, and on what the Universe might have been like earlier than 1000 s.

Suggestions for Further Reading

Steven W. Hawking, *A Brief History of Time,* New York, Bantam Books, 1988.

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